What is knot theory? [see Adams Ch 1].

Study of embeddings of one space in another part of a branch of mathematics called topology, a close relative of geometry. We will consider just one case.

Def: A knot (or knotted circle) is an embedding

\[ K: S^1 \rightarrow \mathbb{R}^3 (\simeq S^3) \]

- \( \mathbb{R}^k = \{ (x_1, \ldots, x_k) \mid x_i \in \mathbb{R} \} \times S^{k-1} = \{ (x_1, \ldots, x_k) \in \mathbb{R}^k \mid \sum x_i^2 = 1 \} \)
- embedding = continuous injection

Often confuse \( K: S^1 \rightarrow \mathbb{R}^3 \) w/ image \( K = K(S^1) \subset \mathbb{R}^3 \) [similarly for \( S^2 \)]

EX

- unknot
- trefoil
- figure eight
We'll be interested in polygonal knots \( K : S^1 \to \mathbb{R}^3 \)
with \( S^1 \) decomposed into finitely many arcs, each of which is sent to a straight segment — unless otherwise stated, all knots are assumed polygonal.

Pictures? Yes, we will draw many pictures and often our proofs will require the use of pictures.

**Def.** A nice projection of a knot \( K \) is the orthogonal projection of \( K \) onto a plane \( P \subset \mathbb{R}^3 \) satisfying:

1. no edge of \( K \) has direction orthogonal to \( P \)
2. the projection is at most 2:1 on \( K \), and is 1:1 on vertices.

\( (\pi : \mathbb{R}^3 \to P \text{ has } \pi'(p) \cap K = \emptyset \text{ at vertices }) \)

**Proposition I.1** Given a knot \( K \), there is a nice projection.

In fact, parameterizing orthogonal projections (up to translation) by unit vectors in \( S^2 \) (up to sign), there are nice projections.
Corresponding to a subset $X \subset S^2$ which is the complement of a finite # of points and arcs.

**Proof:** (1) requires us to omit a finite set of points, namely the unit vectors parallel to the edges of $K$.

(2) 1:1 a vertex $\Rightarrow$ omit directions from a vertex to edges $\subset$ a finite # of planes
planes define great circles on $S^2$

2:1 $\Rightarrow$ omit direction of line going through 3 edges -- finitely many come in $S^2$

**Definition:** A diagram $D$ of a knot $K$ is a nice projection of $K$, together with each crossing (point of noninjectivity) indicated as over/under.

**Equivalence of knots:** Several choices

1. $K$ & $K'$ are ambient isotopic if $\exists$ continuous map

   $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ s.t. $H_t(K) = H_t(K')$ is a homeomorphism $\forall t \in [0,1]$ and $H_0 = \text{id}_{\mathbb{R}^3}$, $H_1(K) = K'$ -- $\exists$ a 1-parameter family of homeomorphisms deforming $K$ to $K'$

   $\Rightarrow$ We often draw diagrams "smoothly" instead of polygonally.
(2) \( K \& K' \) are combinatorially equivalent if \( f \) a sequence of moves
\[ \xymatrix{ \triangle & \square } \]
the defining triangle \( \triangle \) meets knots only along the boundary.

(3) \( K \& K' \) are homeomorphically equivalent if \( f \) an orientation preserving homeomorphism taking \( K \) to \( K' \).

**Theorem 1.2** All three of these relations are the same for knots.

Idea: (1) \( \Rightarrow \) (3) is clear since \( H_1 : \mathbb{R}^3 \to \mathbb{R}^3 \) is an o.p. homeo.

(3) \( \Rightarrow \) (1) Theorem that any o.p. homeo is isotopic to identity.

(1) \( \Rightarrow \) (2) (2) is essentially the "discrete" version of (1). We approximate by "piecewise" then break up result into small pieces. \( \square \)

**Basic problem:** when are two knots equivalent? (we'll just call them "the same") How can we tell?

**Ex**

\[ \begin{array}{c}
\includegraphics{example.png} \\
\end{array} \]

why? - Always go under existing arc of diagram, so if proj. plane is, say, \( (x,y) \)-plane, can assume \( z \)-coordinate decreases until just before closing up. \( \square \)