Last time we defined a knot in $\mathbb{R}^3$ to be a polygonal embedding $K: S^1 \to \mathbb{R}^3$ (or its image $K = K(S^1)$).

A nice projection exists and gives us a diagram of the knot.

Can view $D$ as a graph with vertices of valence 2 (images of vertices of $K$) and vertices of valence 4 (the crossings), the latter marked as over/under.

Also defined equivalence relation on knots (w/ 3 different descriptions).

Exercise III: If $K, K'$ have the same diagram, then $K \sim K'$.

What if the diagrams are different?

Let's look at ways the diagrams could be different:

Planar isotopy:

Suppose $K, K'$ have diagrams $D, D'$ in a plane $\mathbb{R}^2$, say $D, D'$ are related by planar isotopy if $\exists H : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ w/ $H_0 = \text{identity on } \mathbb{R}^2$, $H_1(D) = D'$ respecting over/unders at crossings.
then $K \sim K'$ extend $H$ by identifying a 3rd coordinate in $\mathbb{R}^3$ to a map $\tilde{H}: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$, $\tilde{H}(K) = K$ is a knot with the same projection at $K'$, so $K \sim \tilde{H}(K) \sim K'$.

Reidemeister moves: There are 3 ways to modify the diagram of a knot that clearly does not change the knot type, these are called the Reidemeister moves;

- D,D' $\subset \mathbb{R}^2$ two diagrams, the same outside a disk; differing in the disk by:
  - type I: \[ \begin{array}{c} \text{disk} \quad \leftrightarrow \quad \text{disk} \end{array} \]
  - type II: \[ \begin{array}{c} \text{disk} \quad \leftrightarrow \quad \text{disk} \end{array} \]
  - type III: \[ \begin{array}{c} \text{disk} \quad \leftrightarrow \quad \text{disk} \end{array} \]

- [Exercise I.2: Show that the other Reidemeister moves below are a consequence of these 3 (and planar isotopy).]

Theorem I.3: $K, K'$ two knots an equivalent iff the diagrams $D, D'$ for $K, K'$ differ by a finite sequence of Reidemeister moves I, II, III and planar isotopy.

Idea of proof: Need to see if $K \sim K'$, then $D, D'$ differ as in theorem.

Know $\exists K = K_1, K_2, \ldots, K_n = K'$ w/ $K_i, K_{i+1}$ related by triangle move!
By small change in projection \( \pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) we can assume \( \pi \) is a nice projection for every \( K_i \). [Also note that small change of projection changes \( D \& D' \) to planar isotopic diagrams.]

Further, we can assume \( \pi \) is injective on all triangles defining moves. Now replace each triangle move with finitely many small moves of type I, II, and III. E.g.,

![Diagram showing three moves I, II, and III]  

[Remark: We can essentially take this theorem as definition of equivalence — i.e., Knutill diagrams are related as a thin. This allows us to view knots and question of equiv. of knots as a combinatorial problem.]

How does this help? [replaced existence/non-existence of one type of sequence with another]

We can use this to define knot invariants:
Let \( \text{Knot}(\mathbb{R}^3) \) be the set of all knots in \( \mathbb{R}^3 \).

**Def.** A knot invariant is a function

\[
F : \text{Knot}(\mathbb{R}^3) \to S \quad
\]

where \( S \) is a set with an equivalence relation \( \cong \) on it such that if \( K \cong K' \), then \( F(K) \cong F(K') \).

\( S \) could be a set of groups, abelian groups \( (\cong \text{equivalence}) \), or polynomials, \( \mathbb{R}, \mathbb{Z}, \mathbb{C} \), or trivial relation \( \approx \). Anything!

How do we use Reidemeister moves to do this?

**Ex.** \( \text{Tria} : \text{Knot}(\mathbb{R}^3) \to \mathbb{Z} \)

\( \text{Tria}(K) \) is defined as follows: Let \( D \) be a diagram of \( K \)

\( \text{Tria}(K) = \# \) of ways to color overpassing arcs of \( D \) with 3 colors (using at least \( 2 \)) so that at any crossing, either all 3 colors appear, or exactly one does:

\[
\begin{array}{c}
\text{K = trefoil, } \text{Tria}(K) = 60 \quad \text{(if two arcs are same color, all are)} \\
\end{array}
\]

If \( K \cong K' \), why is \( \text{Tria}(K) = \text{Tria}(K') \)? check invariance under \( \text{RI, II, III} \);

that is, if \( D \cong D' \) by \( \text{RI, II, III} \), then a coloring of one uniquely determines a coloring of the other.
III: orbit-1 correspondence between oarves of $D$ & $D'$

![Diagrams](image)

check:
1. $w, w'$ are determined by $u, v, x, y, z$ - easy ($u, v, w \in \mathbb{S}^1$)
2. If $u, v, x, y, z, w$ is allowable, then $\exists w'$ s.t. $u, v, x, y, z, w'$ is allowable (it's unique by (1)) - cases $x \neq y, z$ exactly 1 possibility
3. If $x = y, z$ then $w, w'$ exactly 1 possibility.

This proves

Theorem 1.4 Tri is a knot invariant

Corollary 1.5 $\mathbb{S}^0 \neq \emptyset$

Links: A link is a collection of pairwise disjoint knots.

Def: A link is a (polygonal) embedding $L: \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{R}^3$. We also write $L = L(S^1)$. The restriction to one of the circles is called a component $L_i: L_i: S^1 \rightarrow \mathbb{R}^3$ (or $L_i(S^1) = L_i$) and each $L_i$ is a knot.

Say $L$ has $k$ components in the case. Can an empty link be a knot?

Some definition of equivalence, and

1.1-1.4 also hold for links.

$k$-component unlink:
Corollary I.6: There are infinitely many, pairwise inequivalent knots.

**Proof:** Consider the knot $K_k$: 

\[ \text{trefoil} \]

Check that there are $3(3^k - 1)$ tricolorings, so 

\[ \text{Tri}(K_k) = 3(3^k - 1) . \]

Therefore, $\text{Tri}(\text{Knot}(\mathbb{R}^3))$ is infinite, hence so is the # of equivalence classes in Knot$(\mathbb{R}^3)$.

**Exercise I.3** There are only a countably infinite number of equivalence classes of knots. (Of course, Knot$(\mathbb{R}^3)$ is uncountable)

Link example: 

\[ W = \text{White head link} \]

Note: $W \neq 00$ since $\text{Tri}(W) = 0$, but $\text{Tri}(00) = 6$. 