This requires a little more story...

A retraction \( r: X \to A \) is a continuous map of a top.
space \( X \) to a subspace \( A \subset X \) st \( r(a) = a \ \forall a \in A \).

**Proposition III.49** If \( x \in A \subset X \) and \( r: X \to A \) is a
retraction then \( r_X: n_x(X, x) \to n_x(A, x) \) is surjective. If \( r_X: n_x(x, a) \to n_x(x, r(x)) \)
is injective, when \( i: A \to X \) is inclusion \( \Box \).

**Proof (exercise)** (special case \& def. retraction \( r_x = (i_x)^{-1} \)). \( \Box \)

**Corollary III.50** If a retraction \( r: \overline{B^2} \to S^1 \).

**Proof**: \( n_1(\overline{B^2}) = \pi_1(S^1) \). \( \pi_1(S^1) \cong \mathbb{Z} \). \( \Box \)

**Proof of BPT**. Suppose \( \exists f: \overline{B^2} \to \overline{B^2} \) w/ no fixed pt.

Define \( r: \overline{B^2} \to S^1 \) as follows. \( \forall x \in \overline{B^2} \), let \( L_x \)
be the line through \( x \) \& \( \text{fix} \). \( \text{WELL DEFINED} \) \( \sin \theta = r_x \).
oriented from \( \text{fix} \) toward \( x \). Define \( r(x) \in S^1 \) to be
the 1st point of intersection of \( L_x \) \& \( S^1 \) st. \( r(x) = x \) on \( L_x \)
— so \( r\text{fix}=x \) if \( x \in S^1 \). Check continuous. This is a
retraction, contradicting III.49 \( \Box \).

Another application is:

**Fundamental Theorem of algebra III.51** Any nonconstant polynomial
\( f(z) \in \mathbb{C}[z] \) has a root in \( \mathbb{C} \).
Proof: WLOG, write \( p(t) = t^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0 \) with \( n \geq 0 \).

Assume \( p(t) \) has no roots in \( \mathbb{C} \). A \( \mathbb{R} \) define

\[
\chi_r(t) = \frac{p(re^{2\pi i t})/p(r)}{1+p(re^{2\pi i t})/p(r)}
\]

which is a loop based at \( 1 \) in \( \mathbb{S} \) for all \( r > 0 \).

Since \( \chi_0(t) \) is constant, \( [\chi_r] = [\chi_0] = \Delta \), \( r > 0 \).

(b/c \( \chi_r \to \chi_0 \) as \( r \to 0 \))

Now consider the 1-parameter family of polynomials

\[
P_s(z) = z^n + \frac{s}{(a_{n-1}z^{n-1} + \ldots + a_1z + a_0)}, \quad s \in [0,1].
\]

This gives loops \( \chi_{r,s} \) defined via \( P_s \) as \( \chi_r = \chi_{r,1} \) is via \( P = P_1 \).

If \( r \) is sufficiently large (at least \( |a_0 + a_{n-1}| + 1 \))

Check: \( P_s \) has no roots on \( \{ z : |z| = r/2 \} \). For any \( s \), then

\[
\chi_{r,s}(t) = e^{r t} \frac{e^{2\pi i s/n}}{e^{2\pi i s/n}} = e^{2\pi i s/n}, \quad [\chi_{r,s}] \neq 0 \text{ in } H_1(\mathbb{S}^1, \mathbb{S})
\]

and since \( \chi_{r,0} \to \chi_{r,1} \), we have a contradiction \( \square \)

... And another...

Borsuk-Ulam Theorem (1=2) III. 52

If \( f : S^2 \to \mathbb{R}^2 \) is continuous, \( \exists x \in S^2 \) \( f(x) = f(-x) \). (in \( f \) identifies a pair \( x, -x \) antipodal)

Proof: Sketch — assume not, set \( g(x) = \frac{(f(x) - f(-x))}{|f(x) - f(-x)|} \) observe \( g : S^2 \to S^1 \) and \( g(-x) = -g(x) \), check that \( g([\text{Equator}]) \) is non-trivial in \( H_1(S^1) \). This contradicts fact [equator] \( = 1 \) in \( H_1(S^2) \) \( \square \)
Proposition III 5.4 \[ \pi_1(X \times Y, (x,y)) \cong \pi_1(X,x) \times \pi_1(Y,y) \]

Proof: Viewing \( X \times Y \) as \( \mathbb{R} \times \mathbb{R} \), the projections \( p_x: X \times Y \to X \) and \( p_y: X \times Y \to Y \) clearly are \( (p_x)_* \times (p_y)_*: \pi_1(X \times Y, (x,y)) \to \pi_1(X,x) \times \pi_1(Y,y) \) and

\[ (g \times h) \mapsto (g \times (h), g \times y) \]

is the map of \( \mathbb{R} \times \mathbb{R} \).

If \( x \leq x_1, y \leq y_1 \), then \( x \times y \leq x_1 \times y_1 \), so \( \ker (p_x)_* \times (p_y)_* \) is trivial \[ \square \]

Corollary III 5.5 \[ \pi_1(T^n) \cong \mathbb{Z}^n \quad \forall n \geq 1 \]

Proof: \( T^n \cong S^1 \times S^1 \times \cdots \times S^1 \), which is a product of \( \pi_1 \).

\[ \pi_1(S^m \times S^n) \cong \pi_1(S^1) \cong \mathbb{Z} \quad \forall m, n \geq 2 \]

Proposition III 5.3 \[ \mathbb{R}^2 \cong \mathbb{R}^n \implies n = 2 \]

Proof: We have already seen that \( \mathbb{R}^n \cong \mathbb{R}^2 \) is homotopy equivalent to \( S^{n-1} \). So \( \mathbb{R}^2 \cong \mathbb{R}^n \implies S^1 \text{ homotopy equivalent to } S^{n-1} \).

If \( n = 1 \), \( S^0 = \{0, \infty\} \), disconnected. \[ \uparrow \]

If \( n > 2 \), \( \pi_1(S^{n-1}) = \{1\} \) \[ \uparrow \]

So, \( n = 2 \). \[ \square \]

Remark: \( S^k \) is disconnected only for \( k = 0 \) \[ \implies \pi_1 \cong \mathbb{Z}^n \implies n = 1 \]
Definition: A map \( p: \tilde{X} \rightarrow X \) is called a covering map if 
\[ \forall x \in X, \exists \text{ a nbhd } U_x \text{ of } x \text{ st } \]
\[ p^{-1}(U_x) = \bigsqcup V_x^\alpha \text{ is a disjoint union of } \tilde{X} \text{ open and } \]
\[ V_x^\alpha \subset \tilde{X} \text{ open and } \]
\[ p|_{V_x^\alpha} : V_x^\alpha \rightarrow U_x \text{ a homeomorphism } \forall x \in X. \]

The key property of covering spaces is homotopy lifting:

**Lemma 3.56** 
If \( p: \tilde{X} \rightarrow X \) is a covering map, \( f: Y \rightarrow X \)

is a map with \( \tilde{f} : Y \rightarrow \tilde{X} \), and \( H: Y \times I \rightarrow X \) \( \text{ a homotopy of } \)

\( f \) rel \( A \), then \( \exists ! \) lift \( \tilde{H}: \tilde{Y} \times I \rightarrow \tilde{X} \) of \( H \) which is a homotopy of \( \tilde{f} \)

rel \( A \).

**Proof:** Same idea as in proof of Lemma 3.56: Exact same proof.

Given \( \exists ! \text{ unique lift } \tilde{f}|_{\tilde{Y} 	imes \{0\}} \).

So, we have \( \tilde{f} \) and \( \tilde{H} \) continuous on \( \tilde{Y} \times I \).

So, we have \( \tilde{H} \) that covers \( f \) rel \( A \).

(Unique by unique lift \( \tilde{f}|_{\tilde{Y} \times \{0\}} \)). \( \Box \)

**Corollary 3.57** 
If \( p: \tilde{X} \rightarrow X \) is covering map, \( f: \tilde{X} \rightarrow Y \), \( \tilde{f} \) \( \text{ is } \)

then \( \exists ! \text{ lift } \tilde{f}: \tilde{X} \rightarrow \tilde{Y} \) if \( H \) is homotopy of \( f \) rel \( \tilde{Y} \).

then \( \exists ! \text{ lift } \tilde{H} \) which is a homotopy of \( \tilde{f} \) rel \( \tilde{Y} \). \( \Box \)

**Proposition 3.58** 
If \( p: \tilde{X} \rightarrow X \) is covering space, \( \tilde{f}: (\tilde{X}, \tilde{x}) \rightarrow (X, \tilde{x}) \) the

\( p \circ (\tilde{f}) \) is injective. \( p_*(\pi_1((\tilde{X}, \tilde{x})) \rightarrow \pi_1((X, \tilde{x})) \)

\( \tilde{f} \) lifts to a loop \( f \) on \( X \) based at \( \tilde{x} \).

**Proof:** Exercise from 3.57. — compare with calculation of \( \pi_1(S^1) \). \( \Box \).
Example 1

\( p : \mathbb{R} \rightarrow S^1 \) is a covering map. Other covering maps of \( S^1 \) are given by \( p_n(z) = e^{2\pi i n} \) (viewing \( S^1 \subset \mathbb{C} \)).

\((p_n)_* : \pi_1(S^1, 1) \rightarrow \pi_1(S^1, 1)\) is given by \((p_n)_*([\gamma_n]) = [\gamma_n]\) when \( \gamma_n(t) = e^{2\pi i nt}, t \in [0, 1] \). That is, w.r.t. \( \pi_1(S^1, 1) \cong \mathbb{Z} \), we have \((p_n)_*([1]) = n \forall n \in \mathbb{Z} \).

2) \( X = \bigwedge_{a}^{\infty} \bigvee_{a}^{\infty} \) is a wedge of \( \mathbb{Z} \) circles.

The following graphs are all covering spaces of \( X \) with covering map that sends a-edges, b-edges homeomorphically onto a-edge/b-edge:

(i)
(ii)
(iii)
(iv)
(v)
(vi)

Z \times \mathbb{R} \sqcup \mathbb{R} \times Z

Infinite regular 4-valent tree