Last time: Given a prime $p \geq 3$, we defined
\[ C_p(K) = \text{vector space of } p\text{-colorings of a diagram of } K \text{ over } \mathbb{Z}_p \]
\[ \dim C_p(K) > 1 \quad (\text{trivial colorings — all the same value}) \]

Define the \textit{mod } $p$ \textit{rank of } $K = \text{rank}_p(K) = \dim C_p(K) - 1$

\[ \text{Exercise I.7} \quad \# \text{ of nontrivial } p\text{-colorings of } K = (\text{rank}_p(K) \cdot p - 1) p \]

Also computed $\text{rank}_p(K)$ from a matrix as follows

\[ \text{D = diagram for } K \text{ with } k \text{ crossings} \]

\[ M_D = \begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 2 \\ -1 & 2 & 0 & -1 \end{pmatrix} \]

\[ x_k - x_i \cdot x_j = x_i + x_j \mod (p) \]

\[ \text{Let } M_D' \text{ be some } (k-1) \times (k-1) \text{ minor (any will do)} \text{ and} \]

\[ \text{then } \text{rank}_p(K) = \dim (\text{null}(M_D')) \]

In particular $\text{rank}_p(K) \neq 0$ iff $p \mid \det(M_D')$

\[ \det(M_D') = 2(3) - 1(1) = 6 - 1 = 5 \]

In Figure 8, $\text{rank}_p(K) = 0$ unless $p = 5$.

\[ \begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -2 & 3 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 & 3 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 & 3 \\ 1 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} \]

So, $\text{rank}_p(K) = 1$. 

Idea was: multiply certain rows by -1, according to black/white checkerboard coloring.

\[ \begin{bmatrix} +1 & -1 \end{bmatrix} \]

then sum of rows = 1.

**Exercise I.8:** If \( A \) is a \( k \times k \) matrix and all rows and columns run to 1, prove that every \((k-1) \times (k-1)\) minor has the same determinant, up to sign, w/ few crossings.

**Data:** If \( D \) is a diagram for \( K \), define \( \det(D) \) to be \( |\det(M_0^0)| \), for any \((k-1) \times (k-1)\) minor \( M_0^0 \) \( \det M_0^0 \times 0 \times 0 = 1 \)

by exercise, this is independent of the choice \( \{ \}\) minor.

By Cor. I.11, the prime divisors of \( \det(D) \) give an \( m \) \( \mid K \), and in fact we have

**Proposition I.12:** If \( D \) is a diagram of \( K \), then \( \det(D) \) is an \( m \) \( \mid K \).

We write \( \det(D) = |\det(K)| \), call this the determinant of \( K \).

**Exercise I.9:** Compute determinant and \( \text{mod } p \) rank for all knots up to 6 crossings (see tables online), and all \( \text{prime } p \geq 3 \).

**Proof of Proposition:** Need to check invariance by Reidemeister moves. [Convenient choice of numbering is helpful.]

**Exercise I.10:** Prove that for knots, there are no nontrivial 2-colorings. **I.11:** Prove \( \det(K) \) is odd \( \forall \) knots \( K \).
In D, label our arc involved $x_1$, let $c_1, c_2, c_3, c_4$ be crossings $x_1$ passes through, in order, so that $c_1, c_4$ before $D$ and $c_2, c_3$ after.

In $D'$, $x_1$ becomes $x_0, x_1'$ and $x_i = x_i'$ for $i > 1$. [Recall diagrams agree]

$C_0 = \text{new crossing, } C_i = C_i'$ for $i > 1$

$$M_D = \begin{bmatrix} -1 & 2 & \ldots & 2 & -1 \\ 2 & 2 & \ldots & 2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & -1 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \end{bmatrix}$$

$$M_{D'} = \begin{bmatrix} 1 & -1 & \ldots & 0 & 0 \\ 2 & 0 & \ldots & 0 & 2 \\ 0 & 2 & \ldots & 2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -1 \\ \end{bmatrix}$$

Add column 0 to column 1 in $M_{D'}$ (don't change det)

Remove last row & column to get $M_{D'}$

Look at conductor expansion of $\det(M_{D'})$ along top row - all terms are zero except $1 \cdot \det(M_0)$

Need to be a little careful --- could have $c_s = c_j$, some $j$:

My similar argument, maybe this is all crossings, special care with low crossing $- \det(0 \times 0 \text{ matrix}) = 1$...
Exercise I.12  Check RII invariance. (Assume, if you like, that picture is with it & best easy distinct.

\[ M_D \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 100 & 0 \\ 0 & 2 & 0 & 0 & 1 & 100 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ M_{D'} \begin{bmatrix} 0 & 0 & 2 & 1 & 0 & 100 & 0 \\ 0 & 2 & 0 & 0 & 1 & 100 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Pick eight minors, 1st row & column, expand along new 5th order using cofactor expression. Same matrices. Special cases again...