**II. Polynomial Invariants**

**Alexander polynomial (1st look)**

There is a refinement of determinant -

Alexander polynomial $\mathcal{A}(K) = A_K(t) \in \mathbb{Z}[t^\pm 1] = \text{Laurent polynomials}$

so $A_K(t) = \sum_{k=-N}^{N} a_k t^k$

We can think of $\mathbb{Z}[t^\pm 1] = \mathbb{Z}[t]^\mathbb{Z}$, when $G = \langle t \rangle \cong \mathbb{Z}$ integral group ring.

Defined as follows:

1. orient the knot - give it a direction
   (2 choices for a knot, 2 more for a reverse knot)

   ![Diagram]

   Construct a matrix with rows for crossings, columns for our arcs, and entries as follows

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1-t$</td>
<td>$t$</td>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$1-t$</td>
<td>$t$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$t$</td>
<td>$-1$</td>
<td>$1-t$</td>
<td></td>
</tr>
</tbody>
</table>

If $x_i = x_j$, $x_j = x_k$, or $x_i = x_k$ (or any 3x), then add up.
Get $M_D(t) \in M_{kk} \left( \mathbb{Z}[t^{\pm 1}] \right)$, $M_D(t)$ any minor of an Alexander matrix for $D$.

$A_D(t) = \det(M_D(t))$ is "the" Alexander polynomial of $D$.

Observe: $|A_D(-1)| = \det(D)$.

**Theorem II.1** If $D, D'$ are diagrams for knots $K, K'$, then $A_D(t) = \pm t^k A_{D'}(t)$ for some $k \in \mathbb{Z}$.

In fact, $A_D(t)$ depends on choice of minor, up to $\pm t^k$ factor.

Proof of this, and that $A_D(t) = \pm t^k A_{D'}(t)$ for oriented diagrams is similar to, but more involved than, minimality of det.

Changing orientation: replace $t$ by $t^{-1}$ (up to $t^k$):

\[
\begin{array}{cccc}
x_i & \rightarrow & x_i \\
\downarrow & & \downarrow \\
-x_i & \\
1-t & -1 & t & 1-t, -1, t \rightarrow 1-t^{-1}, -1, t^{-1} \rightarrow -t+1, t, -1.
\end{array}
\]

So, change orientation $D \rightarrow D'$ get $A_D(t) = \pm t^k A_{D'}(t^{-1})$.

Fact: up to $\pm t^k$, $A_D(t)$ is palindromic: $A_D(t) = \pm t^k A_{D'}(t^{-1})$.

**Exercise II.1** Compute $A_K(t)$ for each knot $K$ with less than 6 crossings in the table.