From now on, assume $B = A^{-1}$, $d = -A^2 - A^{-2}$, so that $\langle d \rangle \in \mathbb{Z}[E(A^{-1})]$.

Lemme II.5  If $D, D'$ are related by $RI$, then $\langle D \rangle = \langle D' \rangle$.
Proof : Exercise II.6

Exercise II.7 Compute $\langle D \rangle \in \mathbb{Z}[E(A^{-1})]$ for all diagrams $D$ with up to 6 crossings in the table.

What about $RI$?  Math 428  2/17/11

Lemme II.6  We have $-A^3 \langle d^- \rangle = \langle -\rightarrow \rangle = -A^3 \langle -\sigma \rangle$

Proof : Exercise II.8.

What to do?  don't want to require $A^3 = -1$...

Fix requires notion of writhe

Define if $D$ is the diagram of an oriented link, the writhe of $D$ is defined to be $\left( \sum_{\text{right handed}} + \sum_{\text{left handed}} \right) =: \omega(D)$

\[
\omega \left( \includegraphics{diagram1} \right) = 3 \quad \omega \left( \includegraphics{diagram2} \right) = 0
\]

Note $\omega(D) = \omega(-D)$. 

How does \( w \) change under \( R \)-moves?

\( R_II, R_{III} \) clearly do not change it.

However,

\[
\begin{align*}
\omega(R) &= 1 + \omega(R) \\
\omega(R') &= -1 + \omega(R)
\end{align*}
\]

**Theorem II.7** If \( D, D' \) are diagrams, for oriented links \( L \sim L' \), then

\[
(-A)^{-3\omega(D)} \langle D \rangle = (-A)^{-3\omega(D')} \langle D' \rangle
\]

**Proof** We can assume \( D \) and \( D' \) differ by \( R \)-move. Since \( w(D) \) \& \( \langle D \rangle \) are unchanged by \( R_{II} \) \& \( R_{III} \) moves, we can assume \( D \) and \( D' \) differ by \( R_I \) move:

\[
D \sim D' \Rightarrow \omega(D') = 1 + \omega(D)
\]

so

\[
(-A)^{-3\omega(D)} \langle D \rangle = (-A)^{-3\omega(D') - 1} \langle D' \rangle = (-A)^{-3\omega(D')} \langle D' \rangle
\]

Define \( V_L(A) = (-A)^{-3\omega(D)} \langle D \rangle \) for any diagram \( D \) of an oriented link.

**Corollary II.8** For any oriented link \( V_L(A) \) is an invariant in \( \mathbb{Z}[A, A^{-1}] \) with trivial equivalence. If \( K \) is a knot, \( V_K(A) \) is an invariant of oriented knots (independent of choice of orientation used to compute).

Define the reflection of a link \( L \) is the link \( L' \) obtained by applying a reflection \( \sigma^I \) \& \( R^I \) - for a diagram, just change all crossings.
Exercise II.9: Prove that $\langle D \rangle (A) = \langle \bar{D} \rangle (A^{-1})$
and $V_L(A) = V_L(A^{-1})$, where $D$ is a diagram of $L$ and $\bar{D}$ is its mirror.

Exercise II.10: Prove that the right-handed and left-handed trefoils are not equivalent.

If $K \neq \bar{K}$, then $K$ is called chiral. If $K = \bar{K}$, then $K$ is called achiral or amphichiral.

Exercise II.11: Prove that figure 8 is achiral $\square$.

Exercise II.12: Compute $V_L(A)$ where $T_{2,n}$ is the right-handed $(2,n)$-torus link, $V_n$.

Can be embedded on standard torus $\mathbb{R}^2$.

$\square$ goes twice around one direction, $n$ times around the other.

Exercise II.13: Prove that $V_L(A)$ is in $\mathbb{Z}[A_1^2, A_2^2]$.

Convention: substitute $t^2$ for $A^2$ making $V_L \in \mathbb{Z}[t^2, t^3]$. (Classical and not always agreed upon) We will see that if $L$ is a knot, then $V_L \in \mathbb{Z}[t, t^2]$ i.e. no $t^3$ powers actually exist.
Proposition II.9. The Jones polynomial \( V_L(t) \in \mathbb{Z}[t^\frac{1}{3}, t^{-\frac{1}{3}}] \) is an invariant of oriented links satisfying:

(i) \( V_\emptyset(t) = 1 \), where \( \emptyset \) is oriented unknot.

(ii) Given 3 links \( L^+, L^-, L \), which are the same outside a ball, and inside satisfy

\[
\begin{array}{c}
\begin{array}{c}
\stackrel{L^+}{\rightarrow} \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\stackrel{L^-}{\rightarrow} \\
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
\stackrel{L}{\rightarrow} \\
\end{array}
\end{array}
\}

then

\[
t^{-1} V_{L^+}(t) - t V_{L^-}(t) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) V_L(t) = 0.
\]

Proof: 1st check behavior at bracket (or let \( L \) denote link &)

\[
\begin{align*}
\langle X \rangle &= A \langle > \rangle + A^{-1} \langle = \rangle \\
\langle X' \rangle &= A^{-1} \langle > \rangle + A \langle = \rangle,
\end{align*}
\]

\[\Rightarrow A \langle X \rangle - A^{-1} \langle X' \rangle = (A^2 - A^{-2}) \langle > \rangle \iff A \langle L^+ \rangle - A^{-1} \langle L^- \rangle = (A^2 - A^{-2}) \langle L \rangle,
\]

observe that writhe are related by

\[
\omega(L^+) = \omega(L) \pm 1,
\]

so

\[
(A^2 - A^{-2}) V(A) = (A^2 - A^{-2})(-A)^{-3\omega(L)} \langle L \rangle
\]

\[= (-A)^{-3\omega(L)}(A \langle L^+ \rangle - A^{-1} \langle L^- \rangle)
= A(-A)^{-3\omega(L) + 1} (A^3 \langle L^+ \rangle - A^{-1}(-A)^{-3(3\omega(L) - 1)} (-A)^3 \langle L^- \rangle
= -A^q V_{L^+}(A) + A^{-q} V_{L^-}(A), \quad \text{substitute } t^{\frac{1}{2}} = A^{-2} \quad \square \]
Proposition II.10 If \( L \) is an oriented link with an odd number of components, then \( V_L(t) \in \mathbb{Z}[t, t^{-1}] \)

If \( L \) has an even number of components, then \( V_L(t) \in \mathbb{Z}[t^{\pm 1}] \)
and has only \( k \)-integer powers.

Proof: Induction on \# of crossings in a diagram. (Again, consider diagram 8 (a/c)

Base case: 0 crossings, then if \( L \) has \( k \) components, no crossings \( w(L) = 0 \) and

\[
V_L(A) = \langle L \rangle = (\lambda A^2 - \lambda^{-2})^{k-1} = (-1)^{k-1} A^{2(k-1)} (1 + A^{-2})^{k-1}
\]

Substituting \( t^{\pm 2} = A^{\pm 2} \) we get

\[
V_L(t) = (-1)^{k-1} t^{-\frac{2(k-1)}{2}} \left( \frac{1}{1+t} \right)^{k-1}
\]

Introduce all \( k \)-int. powers if \( k \) even. ✓

Support true for up to \( n \) crossings, prove for \( n+1 \): –

\( \checkmark \) Observe that by changing some of the crossings we can make any diagram into the diagram of a unknot, 0000.

To go from \( n \) to \( n+1 \) crossings, we induct on the number of crossing changes needed to change the diagram into the diagram of an unknot.

\( \checkmark \) Compare with example at end of 1st day.
The basic case here is that we require 0 crossing changes. Then because $V_L$ is a link and, result follows from inductive hypothesis of initial induction. Suppose we know it's true now for diagrams with at most $k+1$ crossings requiring at most $k$ crossing changes to make the diagram that of the unlink. Let $L_+$ be a diagram with $n+1$ crossings requiring $k+1$ crossing changes to make $L_+$ into the unlink. Pick one of the $k+1$ crossings and let $L_-$ be the link obtained by changing that crossing.

Then $L_-$ requires $k$ crossing changes to make it into the unlink, hence the result holds for $L_-$. Now by previous proposition we have

$$t^{-1} V_{L_+}(t) = t V_{L_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_L(t) = 0$$

or

$$V_{L_+}(t) = t^2 V_{L_-}(t) + \left( t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) V_L(t)$$

$$= t^2 V_{L_-}(t) + t^{\frac{1}{2}} (t - 1) V_L(t)$$

Further note that $|L_+| = |L_-| = |L| + 1$ and the number of crossings of $L$ is $n$, so result holds for $L_+$. Hence result holds for $L_+$ to see this, for example suppose $|L_+|$ is odd, then so is $|L_-|$, while $|L_+|$ is even. From
This, note that \( V_L(t) \) involves only \( k \)-integer powers of \( t \) and hence
\[
t^k(t-1)V_L(t) \text{ involves only integer powers of } t.
\]
Also, \( t^2V_L(t) \) involves only integer powers of \( t \), since \( V_L(t) \) does,
so \( V_L(t) \) involves only integer powers of \( t \), as required. \( \Box \).

The proof of this proposition also proves

**Proposition II.11.** \( V_L(t) \) is determined by the two properties
listed in Prop II.10, and the fact that \( V_L(t) \) is an oriented link invariant.

**Proof sketch.** 1st check that \( V_L(t) = (-t^{k-1} + t^k)^{-1} \) for the
\( k \)-component unlink \( L \) follows from properties in Prop II.10; induct
on \( k \):

\[
\begin{align*}
V'_L(t) - tV_L(t) &= (t^{k-1} - t^k) V_L(t) \\
\frac{t^2 - t}{t^{k-1} - t^k} V_L(t) &= V_L(t) \\
\frac{t^2(1-t^k)}{t-1} &\left( -t^{k-1} + t^k \right)^{-1} \\
(-t^{k-1} + t^k)(-t^{k-1} + t^k)^{-1} &> 0
\end{align*}
\]

This is the base case for induction on \# of crossings, as before
its a double induction, with 2nd induction in \# of crossings required
so change into unlink. \( \Box \)