Open problem. Does there exist $K$ a knot with $V_K(t) = 1$?

$V_K(t)$ is not a complete invariant; $\exists K \neq K'$ s.t. $V_K(t) = V_{K'}(t)$

constructed from mutation as follows.

Let $D$ be a diagram for $L$ and suppose $\exists$ a disk $\Delta$ in projection plane with boundary circle meeting $D$ in 4 points.

![Diagram](image)

Kinoshita-Terasaka knot

Conway knot. [May need to change orientation in strands $\Delta$]

Proposition II.12: If $L'$ is obtained from $L$ by mutation, then $V_L(t) = V_{L'}(t)$.

Proof sketch: Using Prop II.9, express $V_L(t)$ as $\mathbb{Z}t^m, t^n$-linear combination of polynomials $V_{L_1}(t), \ldots, V_{L_r}(t)$ s.t. each $V_{L_i}(t)$ in $\Delta$ looks like 

possibly together with some disjoint circles.

This is achieved by iteratively changing crossings and resolving crossings within $\Delta$. Now check that if we do the same to $L'$, get $L'_1, \ldots, L'_r$ with $\text{and applying Prop II.9, } L_i = L'_i$, $V_{L_i}(t)$ same linear combination of
$V_L(t) \to V_L(t)$ — this is just because the crossing changes and resolutions commute with the rotation. Therefore $V_L(t) = V_L(t)$.

Exercise II. 14. Check this for the Kishino-Terasaka knot and Conway knot.

Applications 8 | Jones polynomial —

Define: Given a link $L$, the crossing number of $L$ is the min. 
$\#$ of crossings in any diagram, write $C(L) = \text{crossing number of } L$. Clearly, $C(L)$ is a link invariant, and $C(L) = 0$ iff $L = \text{unknot}$. Great invariant, except generally impossible to compute.

Define: The diagram $D$ of a link $L$ is called alternating if the crossings along any component of $L$ alternate over/under starting at any point.

Observe that for any projection one can choose crossings so that diagram becomes alternating. To see this, checkboard color diagram, then choose crossings so that $\ast$ occurs at each crossing (instead of $\ast\star\ast\star$).

Exercise II.15. Check that resulting diagram is alternating.
An alternating diagram is "the opposite" behaviour to the unknot construction of always going under. One might expect/hope that alternating knots are "very knotted", & they are, with some obvious exceptional examples:

\[ \bigcirc \bigcirc \bigcirc \bigcirc \sim \emptyset \]

alternating

\[ \bigcirc \bigcirc \bigcirc \bigcirc \sim \bigcirc \bigcirc \bigcirc \bigcirc \]

alternating

Can always add RI move (or reflection) and keep alternating property.

A crossing like this is called a nugatory crossing.

The rest of diagram lies in boxes.

Defn A diagram is called reduced if there are no nugatory crossings.

Theorem II.13 If \( L \) is a link with a reduced alternating diagram with \( n \) crossings, then \( c(L) = n \).
This theorem (proven by Kauffman, Murasugi, Thistlethwaite) answered an inherent question posed in late 1980's!

Try to prove it without Jones polynomial! 

Before proof, need some preliminary lemmas.

Recall a state $S$ of a diagram $D$ is a choice of $A \& \bar{A}$ resolution of each crossing. Number the crossings $1, \ldots, n$. Since $R-A^{-1}$, can view $S$ as a map $S: \{1, \ldots, n\} \rightarrow \{\pm 1\}$, so that the resolution of $i$th crossing determined by $S$ is $A^{S(i)}$. Recall $|S| = \# \text{ components}$ after doing all resolutions.

Proposition II.3 says

$$<0> = \sum_{S} (A^{-1} (-A^{2} - A^{-2})^{n} - 1) = \sum_{S} <\text{DIS}>(-A^{2} - A^{-2})^{1 |S| - 1}$$

Let $S_{+}, S_{-}$ be "constant" states—all $A$-resolutions $\forall$ all $A^{-1}$-resolutions, respectively.

So $S_{\pm}(i) = \pm 1 \forall i$.

Define say $D$ is adequate if $|S_{+}| > |S|$ $\forall$ states $S$ with $\sum S(i) = n - 2$ (so, a state with just one $A^{-1}$ resolution).

Similarly, $D$ is inadequate if $|S_{-}| > |S|$ $\forall$ states $S$ with $\sum S(i) = 2 - n$. $D$ is adequate if $\forall$ both.

Lemma II.14 A reduced alternating diagram $D$ adequate.

proof: $1^{st}$ observe that $D$ is adequate iff each component of $D$ is adequate, so assume $D$ is connected.
Now, when is a diagram adequate? To understand this, first do all resolutions for $S^4$. Then you want to check whether switching any single resolution will result in more or fewer components — note: it either goes up or down, it can't stay the same.

Now, checkerboard color diagram, observe that changing colors if necessary, every black region looks like this. [Could also have "outside" all black]

So, all A-resolution result in circles that bound black regions. How can switching a resolution result in more components?
region can "bump onto itself".

this means we have a nugatory crossing, and this
contradicts assumption D is reduced. So, D is inadequate.

Similar argument shows D is inadequate. □

Exercise II.10 Prove that the pretzel knot diagrams
\[ P(p_1, p_2, q_1, \ldots, q_s) \] are adequate if \( p_i \geq 2, q_i \leq -2, \)
\( r \geq 2, s \geq 2 \)

\[ P(a_1, \ldots, a_n) \]

Exercise II.17 For what values of \( \{a_1, \ldots, a_n\} \) is this
a knot?