

# Differential Geometry: Levi-Civita connection

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**Theorem 6.5** *If  $g$  is a Riemannian manifold, then there exists a unique affine connection  $\nabla$  that is symmetric and compatible with  $g$ .*

*Proof.* If  $\nabla$  exists, then

$$\begin{aligned}\xi(g(\eta, \zeta)) &= g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta) \\ \eta(g(\zeta, \xi)) &= g(\nabla_\eta \zeta, \xi) + g(\zeta, \nabla_\eta \xi) \\ \zeta(g(\xi, \eta)) &= g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta)\end{aligned}$$

Adding the first two lines and subtracting the third gives

$$\begin{aligned}\xi(g(\eta, \zeta)) + \eta(g(\zeta, \xi)) - \zeta(g(\xi, \eta)) \\ = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta) + g(\nabla_\eta \zeta, \xi) + g(\zeta, \nabla_\eta \xi) - g(\nabla_\zeta \xi, \eta) - g(\xi, \nabla_\zeta \eta)\end{aligned}$$

Manipulating the right-hand side and applying symmetry we obtain

$$\begin{aligned} &= g(\nabla_\xi \eta, \zeta) + g(\zeta, \nabla_\eta \xi) + (g(\nabla_\xi \eta, \zeta) - g(\nabla_\xi \eta, \zeta)) \\ &\quad + g(\eta, \nabla_\xi \zeta) - g(\nabla_\zeta \xi, \eta) + g(\nabla_\eta \zeta, \xi) - g(\xi, \nabla_\zeta \eta) \\ &= 2g(\nabla_\xi \eta, \zeta) + g(\nabla_\eta \xi - \nabla_\xi \eta, \zeta) + g(\nabla_\xi \zeta - \nabla_\zeta \xi, \eta) + g(\nabla_\eta \zeta - \nabla_\zeta \eta, \xi) \\ &= 2g(\nabla_\xi \eta, \zeta) + g([\eta, \xi], \zeta) + g([\xi, \zeta], \eta) + g([\eta, \zeta], \xi)\end{aligned}$$

Now we can solve for the only term involving  $\nabla$

$$\begin{aligned}g(\nabla_\xi \eta, \zeta) &= \frac{1}{2} ( \xi(g(\eta, \zeta)) + \eta(g(\zeta, \xi)) - \zeta(g(\xi, \eta)) ) \\ &\quad - g([\eta, \xi], \zeta) - g([\xi, \zeta], \eta) - g([\eta, \zeta], \xi)\end{aligned}$$

This uniquely determines  $\nabla$ , so  $\nabla$  is unique.

On the other hand this last equation serves to **define**  $\nabla$ . □

If we look in local coordinates  $x^1, \dots, x^n$ , and let  $\xi = \frac{\partial}{\partial x^i}$ ,  $\eta = \frac{\partial}{\partial x^j}$  and  $\zeta = \frac{\partial}{\partial x^k}$ , then the formula at the end of the proof becomes

$$\begin{aligned}g \left( \sum_\ell \Gamma_{ij}^\ell \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \right) &= \frac{1}{2} \left( \frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} \right) \\ \sum_\ell \Gamma_{ij}^\ell g_{\ell k} &= \frac{1}{2} \left( \frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} \right)\end{aligned}$$

If we view  $(g_{ij})$  as a matrix, then because inner products are positive (in particular, nondegenerate), there is an inverse  $(g^{ij})$ , i.e.  $\sum_k g_{\ell k} g^{km} = \delta_\ell^m$ . So

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left( \frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij} \right) g^{km}$$