

Math 519, Homework 1

1. Give \mathbb{R}^m the euclidean metric and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be an immersion. Write $f = (f_1, \dots, f_m)$ with $f_i = y_i \circ f$ where y_1, \dots, y_m are the standard coordinates on \mathbb{R}^m . Let $g = f^*(g_{std})$, and find the functions g_{ij} for g in the standard coordinates x_1, \dots, x_n on \mathbb{R}^n , expressed in terms of the partial derivatives of f_1, \dots, f_m .

2. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold and Γ a group. If $\Gamma \times \widetilde{M} \rightarrow \widetilde{M}$ is a properly discontinuous free action, prove that the quotient $M = \widetilde{M}/\Gamma$ can be given the structure of a smooth manifold in such a way that the obvious quotient

$$f : \widetilde{M} \rightarrow M$$

is a local diffeomorphism. Show that if \widetilde{M} is orientable and the action is by orientation preserving diffeomorphisms, then M is orientable. Prove that if Γ acts by isometries, then there exists a Riemannian metric g on M so that $f^*(g) = \widetilde{g}$.

3. Let \mathbb{Z}^n act on \mathbb{R}^n by

$$z \cdot x = 2\pi z + x$$

for all $z \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. Prove that this is properly discontinuous and free. Further show that the action is by isometries with respect to the euclidean metric on \mathbb{R}^n . Find an isometry from $\mathbb{R}^n/\mathbb{Z}^n$ to the flat n -torus.

4. Ch. 1, problem 7 DoCarmo.

5. Suppose G is a Lie group acting transitively on a manifold M : for all $m, p \in M$, there exists $g \in G$ with $g \cdot m = p$. Such an M is called a *homogeneous space*. Suppose further that G has compact point stabilizers, which is to say

$$\{g \in G \mid g \cdot m = m\}$$

is compact (by the first assumption, this is independent of the point m). Prove that there exists a G -invariant Riemannian metric on M , i.e. a Riemannian metric for which the action of G is by isometries. Is this true without the assumption that point stabilizers are compact?

6. Consider the Lie group $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$. As usual, we can identify the Lie algebra, $\mathfrak{gl}_n(\mathbb{R})$ of left invariant vector fields on $GL_n(\mathbb{R})$, with the tangent space at the identity $M_n(\mathbb{R})$ as vector spaces. If we let $\{x_{ij}\}$ denote standard coordinates on $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$, then prove that this isomorphism

$$M_n(\mathbb{R}) \rightarrow \mathfrak{gl}_n(\mathbb{R})$$

is given by

$$A \mapsto \xi_A = \sum_{i,j,k=1}^n x_{ik} A_{kj} \frac{\partial}{\partial x_{ij}}.$$

Prove that the bracket of matrices $A, B \in M_n(\mathbb{R})$ defined by

$$[A, B] = AB - BA$$

makes $M_n(\mathbb{R})$ into a Lie algebra.

Finally, prove that the isomorphism above is in fact an isomorphism of Lie algebras. That is, prove that

$$[\xi_A, \xi_B] = \xi_{[A, B]}.$$