Math 519, Homework 2

1. Given two vector fields $\xi, \eta \in \mathfrak{X}(\mathbb{R}^n)$, viewing ξ as a map $\xi : \mathbb{R}^n \to \mathbb{R}^n$ via the canonical identification $T_m\mathbb{R}^n = \mathbb{R}^n$, prove that
$$\nabla_\eta \xi(m) = (d\xi)_m(\eta_m)$$
satisfies
   (a) $\nabla_{f\zeta+\eta}\xi = f\nabla_\zeta \xi + \nabla_\eta \xi$,
   (b) $\nabla_\eta(f\zeta + \xi) = \eta(f)\zeta + f\nabla_\eta \zeta + \nabla_\eta \xi$,
   (c) $\nabla_\eta \xi - \nabla_\xi \eta = [\eta, \xi]$,
   (d) $\xi(g(\eta, \zeta)) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)$

   That is, prove that ∇ as defined above is the Levi-Civita connection on $\mathbb{R}^n$.

2. Do problems 1–3 and 8 from Chapter 2 of Do Carmo.

3. Let $\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, B_{n,1})$ denote Lorentz space, $\mathbb{H}^n \subset \mathbb{R}^{n,1}$ hyperbolic n–space and ∇ denote the Levi-Civita connection on $\mathbb{H}^n$. Write $\Pi_m : \mathbb{R}^{n,1} \to T_m H^n$ for the $B_{n,1}$–orthogonal projection
$$\Pi_m(v) = v + B_{n,1}(v, m)m$$
where we view m as both a point in $\mathbb{H}^n$ and as a vector in $\mathbb{R}^{n,1}$. Prove that for any two vector fields $\xi, \eta \in \mathfrak{X}(\mathbb{H}^n)$ we have
$$\nabla_\xi \eta\,(m) = \Pi_m((d\eta)_m(\xi_m))$$
where on the right hand side, we view η as a map $\eta : \mathbb{H}^n \to \mathbb{R}^{n,1}$.

4. Do problems 2, 3, 5 from Chapter 3 of Do Carmo.

5. Let $\mathbb{S}^n$ denote the unit n–sphere, and let $m \in \mathbb{S}^n$ and $v \in T^1_m\mathbb{S}^n$. Prove that
$$\gamma(t) = \cos(t)m + \sin(t)v$$
is a unit speed geodesic through m tangent to v. Consequently, the exponential map is given by
$$\exp(v) = \cos(|v|)m + \sin(|v|)v/|v|$$
for every $v \in T_m\mathbb{S}^n$, $v \neq 0$.

6. Let $\mathbb{H}^n \subset \mathbb{R}^{n,1}$ denote hyperbolic n–space and let $m \in \mathbb{H}^n$ and $v \in T^1_m\mathbb{H}^n$. Prove that
$$\gamma(t) = \cosh(t)m + \sinh(t)v$$
is a unit speed geodesic through m tangent to v. Consequently, the exponential map is given by
$$\exp(v) = \cosh(|v|)m + \sinh(|v|)v/|v|$$
for every $v \in T_m\mathbb{H}^n$, $v \neq 0$.