

Math 519, Homework 4

1. Do the following problems from DoCarmo: 2, 6, 7 from Chapter 5, 2, 7, 12 from Chapter 7, and 5 from Chapter 9.
2. Consider \mathbb{R}^n with the Euclidean metric $g_{\mathbb{E}}$ and let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ be the distance to the origin, $\rho(x) = |x|$. Away from 0, ρ is smooth, and $d\rho^2 = d\rho \otimes d\rho$ is a symmetric 2-tensor. Furthermore, $g_{\mathbb{E}} - d\rho^2$ is also a symmetric 2-tensor, and we write

$$g_{\mathbb{S}} = \frac{1}{\rho^2}(g_{\mathbb{E}} - d\rho^2).$$

(Observe that $g_{\mathbb{S}}$ is not a Riemannian metric because it is not positive definite.) If \mathbb{S}_r is the sphere of radius $r > 0$ centered at 0, prove that

$$g_{\mathbb{S}}|_{\mathbb{S}_r} = \frac{1}{r^2}g|_{\mathbb{S}_r}.$$

Prove that $d\rho^2 + g_{\mathbb{S}}$ is a metric on $\mathbb{R}^n - \{0\}$ and that

$$f : \mathbb{R}^n - \{0\} \rightarrow (0, \infty) \times \mathbb{S}^{n-1}$$

given by $f(x) = (|x|, x/|x|)$ is an isometry (where the target is given the product metric with the obvious metrics on the factors).

3. With notation as in the previous problem, note that on $\mathbb{R}^n - \{0\}$, $g_{\mathbb{E}} = d\rho^2 + \rho^2 g_{\mathbb{S}}$.

Now let (M, g) be a Riemannian manifold of constant curvature K , $m \in M$ and $F : B_{\epsilon}(0) \rightarrow B_{\epsilon}(m)$ the inverse of normal coordinates for some ϵ , where $B_{\epsilon}(0) \subset \mathbb{R}^n$ and $B_{\epsilon}(m) \subset M$. So, F is a diffeomorphism sending radial lines from 0 isometrically to radial geodesics from m . Prove that

$$F^*(g) = d\rho^2 + f_K(\rho)g_{\mathbb{S}}$$

where

$$f_K(\rho) = \begin{cases} \frac{\sin^2(\rho\sqrt{K})}{K} & \text{if } K > 0 \\ \rho^2 & \text{if } K = 0 \\ \frac{\sinh^2(\rho\sqrt{-K})}{-K} & \text{if } K < 0 \end{cases}$$

Hint: Use the construction of Jacobi fields for constant curvature manifolds given in Lecture 16 and their description as $(d\exp_m)_{tv}(tw)$ for appropriate $v, w \in T_m M$.

As a consequence of this, prove that if M and N both have constant curvature K , then for any $m \in M$, $p \in N$, there exists $\epsilon > 0$ and an isometry $f : B_{\epsilon}(m) \rightarrow B_{\epsilon}(p)$. That is, any two manifolds of constant curvature K are locally isometric.