So, for manifolds w/ positive sectional curvature $K(\sigma) \geq K_0 > 0$, sufficiently long geodesics (at least $\pi/\sqrt{K_0}$) are not locally minimizing. Sharp for S^n. [We'll see that Ricci curvature bound, in fact suffices]

In order to make this very useful, need to restrict to manifolds for which any two points can be connected by a geodesic

Lecture 19

Hopf-Rinow & Hadamard Thm — Ch 7 DC

To use curvature hypotheses to study a Riemannian mfd, we must impose some further assumptions — If (M, g) is Riemannian mfd, any curvature assumptions on M are automatically valid for $M - A$, where $A \subset M$ is any closed subset.

For example, assuming M is compact, connected, eliminates this type of phenomena.

This is too restrictive for many applications — E.g. the universal covering of a compact manifold need not be compact.

A weaker hypothesis, completeness, enjoys many of the same properties, and is applicable in more situations (as we'll see).

Assume from now on (unless o.w. stated), mfds are connected

(M,g) connected Riem. mfd. is <u>geodesically complete</u> if $\forall m \in M$, $\exp_m$ is defined on all $T_m M$.

Theorem (Hopf-Rinow) (M,g) Riem. mfd, $m \in M$. The following are equivalent.

(a) $\exp_m$ is defined $\forall m \in M$.

(b) $K \subset M$ is compact iff K is closed and bounded

(c) M is complete

(d) M is geodesically complete.

(e) $M = \bigcup_{j=1}^{\infty} K_j$ w/ $K_j \subset K_{j+1}$ cpct sets $\forall j$, $d(m, M-K_j) \to \infty$ as $j \to \infty$.

Furthermore, any of the above imply (f) $\forall\, p \in M$, $\exists$ geodesic $\gamma : [0,a] \to M$, $\gamma(0) = m$, $\gamma(a) = p$ $d(m,p) = \ell(\gamma)$.

<u>proof</u>: First we prove <u>(a) ⇒ (f)</u>.

What direction should we shoot off a geodesic from m to hit p?

Let $\overline{B}_\varepsilon(m)$ be a closed normal ball about m, $q_0 \in \overline{B}_\varepsilon(m)$ closest to p — i.e. $d(q_0,p) \le d(q,p)$ $\forall q \in \overline{B}_\varepsilon(m)$. Clearly, either $p \in \overline{B}_\varepsilon(m)$ and $p = q_0$ and we're done, or else $q_0 \in S_\varepsilon(m)$:= normal sphere.

Then $q_0 = \exp_m(sv)$ for some $v \in T_m M$, $|v| = 1$, $s>0$. Set $r = d(m,p)$.

<u>Claim</u>: $p = \exp_m(rv)$. — v is the direction we should use.

Note, this is true iff

$$\Delta = \{ t \in [0,r] \mid d(\gamma(t), p) = r - t \} = [0,r]$$

where $\gamma(t) = \exp_m(tv)$

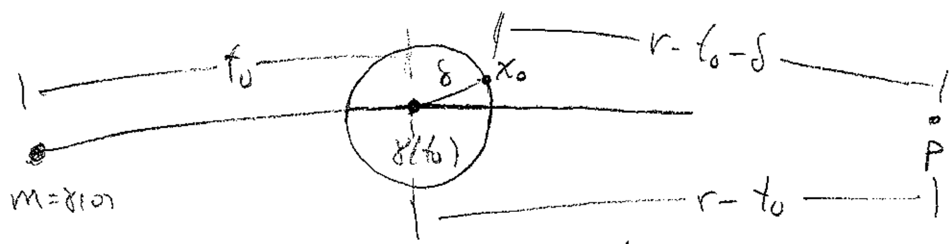
We show that Δ is open & closed, and since $0 \in \Delta$ and $[0,r]$ is connected, we will have $\Delta = [0,r]$.

Clearly Δ is closed since

$$t \mapsto \left(d(\gamma(t), p) - r + t\right)$$

is continuous and Δ = preimage of 0.

Suppose $t_0 \in \Delta$, $t_0 < r$ (If $t_0 = r$, then done). Let $\overline{B}_\delta(\gamma(t_0))$ be a normal ball about $\gamma(t_0)$, $\delta < r - t_0$



Let $x_0 \in S_\delta(\gamma(t_0))$ be a closest point to p in $S_\delta(\gamma(t_0))$

Note: $d(\gamma(t_0), x_0) + d(x_0, p) = d(\gamma(t_0), p)$: Any path from $\gamma(t_0)$ to p can be replaced by one w/ length no longer for which the init segment is a radial geod. ~~from~~ $\gamma(t_0)$ to $S_\delta(\gamma(t_0))$

$$\therefore \quad r = d(m,p) \le d(m, x_0) + d(x_0, p) \le d(m, \gamma(t_0)) + d(\gamma(t_0), x_0) + d(x_0, p)$$
$$= t_0 + \delta + r - t_0 - \delta = r$$

so $\le$ are $=$ & $d(m, x_0) = d(m, \gamma(t_0)) + d(\gamma(t_0), x_0)$

$\therefore$ the concatenated path $\gamma|_{[0,t_0]} \cdot \rho$ where ρ is the geod. from $\gamma(t_0)$ to x_0

is length minimizing $\Rightarrow$ it's a geodesic $\Rightarrow \gamma|_{[0,t_0]} \cdot \rho = \gamma|_{[0, t_0+\delta]}$

So, Δ is open ✓

$\underline{(a) \Rightarrow (b)}$ $K \subset M$ closed & bounded

$\Rightarrow K \subset \exp_m(\underbrace{\overline{B}_r(0)}_{\text{cpct so}}) = \overline{B}_r(m)$ (by (f))

cpct. K closed subset of cpct $\Rightarrow$ K cpct.

converse is true in any metric space ✓

$\underline{(b) \Rightarrow (c)}$ Any Cauchy sequence is bounded and its closure is closed & bounded $\Rightarrow$ the Cauchy seq. has a convergent subsequence —

$\underline{(c) \Rightarrow (d)}$ Need to show that geodesics are defined $\forall$ time t.

Suppose not: $\gamma : (a,b) \to M$, max. unit speed geodesic. wlog suppose $b < \infty$. $\hat{\gamma}$

Note $\{\gamma(b-\frac{1}{n})\}_{n=1}^{\infty}$ is a Cauchy seqn and so converges to some m.

Let W be a δ-totally normal nbhd of m. $N > 0$ s.t. $\forall n > N$

$\gamma([b-\frac{1}{n}, b)) \subset W$. If $\frac{1}{n} < \delta$, then the geodesic $\gamma|_{[b-\frac{1}{n}, b)}$

can be extended to $\gamma|_{[b-\frac{1}{n}, b-\frac{1}{n}+\delta)}$ & $\therefore$ γ can be extended to

$(a, b-\frac{1}{n}+\delta) \supsetneq (a,b)$ $\Downarrow\!\!\Uparrow$

$\underline{(d) \Rightarrow (a)}$ obvious from def'n. ✓

$\underline{(b) \Leftrightarrow (e)}$ true for any metric space □

__Corollary__: M compact $\Rightarrow$ M complete.

__Corollary__: $p : \tilde{M} \to M$ a connected covering space. Then M is complete $\Leftrightarrow$ $\tilde{M}$ is complete

__proof__ see homework □.

__EX__ $\mathbb{R}^n$, $\mathbb{H}^n$, S^n are complete since they are homogeneous - see homework

as are any quotients by isometry groups acting freely and prop disc.

__EX__ G a Lie group w/ a left (right) invt metric — also homogeneous

<u>Theorem</u> (Hadamard). Let (M^n, g) be a complete simply connected Riem. mfld w/ sectional curv. $K(\sigma) \leq 0 \ \forall \sigma$. Then $M \cong \mathbb{R}^n$ — in fact $\exp_m : T_mM \to M$ is a diffeo.

<u>Proof</u> We have already seen that $F = \exp_m : T_mM \to M$ is a local diffeo. Induce on T_mM a Riem. metric by $F^*(g)$. Note that the lines through O_m are biinfinite geodesics — geodesics defined $\forall$ time.

$\therefore$ by Hopf Rinow Thm (a) $\Rightarrow$ (d), we see that (T_mM, F^*g) is complete, and $F : T_mM \to M$ is a local isometry (by construction).

Given $m \in M$, suppose $B_\delta(m)$ is a normal ball about m, observe that $\forall \ x \in F^{-1}(m)$, we have — we used completeness to guarantee $\exp_x$ is defined on $B_{\delta/2}(O_x)$ <u>independent of x</u>

$$\begin{tikzcd} B_{\delta/2}(O_x) \arrow[r, "dF_x"] \arrow[d, "\exp_x"'] \arrow[dr, phantom, "\circ"] & B_{\delta/2}(O_m) \arrow[d, "\exp_m"] \\ B_{\delta/2}(x) \arrow[r, "F"'] & B_{\delta/2}(m) \end{tikzcd}$$

Since dF_x, & $\exp_m|_{B_{\delta/2}(O_m)}$ are 1-1, $F|_{B_{\delta/2}(x)}$ is 1-1.

<u>Claim 1</u> $\forall \ x, x' \in F^{-1}(m)$, $B_{\delta/2}(x) \cap B_{\delta/2}(x') = \emptyset$

<u>Proof</u> If not, $\exists$ geod. from x to x' w/ length $< \delta$. $\gamma : [0,1] \to T_mM$ $\gamma(0) = x$, $\gamma(1) = x'$ $\ell(\gamma) < \delta$

$F \circ \gamma : [0,1] \to M$ is a geodesic of length $< \delta$ w/

$F\circ\gamma(0) = F(x) = m = F(x') = F\circ\gamma(1)$

but this contradicts the fact that $B_\delta(m)$ is a normal ball.

Claim 2. If $y \in F^{-1}(B_{\delta/2}(m))$, then $\exists! x \in F^{-1}(m)$ s.t. $y \in B_{\delta/2}(x)$

— Consequently, $F^{-1}(B_{\delta/2}(m)) = \bigsqcup_{x \in F^{-1}(m)} B_{\delta/2}(x)$ and F is a covering map.

proof of Claim 2: the geodesic from $F(y)$ to m, $\gamma:[0,1] \to M$ lifts to T_mM, $\tilde\gamma:[0,1] \to T_mM$ by $\tilde\gamma(t) = \exp_y(t\,dF_y^{-1}(\gamma'(0)))$ (which makes sense b/c T_mM is complete. $\square$.

Lecture 20

Corollary the universal covering of (M,g) complete w/ $K(\sigma) \le 0$ is $\cong \mathbb{R}^n$.

Ex: (M,g) complete Riem mfld w/ constant curvature $K_0 \le 0$.

$$\mathbb{R}^n \xrightarrow[\text{isom}]{\text{linear}} T_mM \xrightarrow[\exp]{} M$$
$$F$$

F is the univ covering.

Homework exercise (using explicit form of Jacobi field in const curvature)

$$F^*(g) = d\rho^2 + f_{K_0}(\rho)\, g_S$$

where $\rho(x) = d(x,0)$, $g_S = \frac{1}{\rho^2}(g_E - d\rho^2)$ and

$$f_{K_0}(\rho) = \begin{cases} \rho^2 & \text{if } K_0 = 0 \\ \dfrac{\sinh^2(\rho\sqrt{-K_0})}{-K_0} & \text{if } K_0 < 0. \end{cases}$$

Theorem Suppose (M,g) has constant curvature K_0 and is complete.

- If $K_0 = 0$, then the universal cover of M is Euclidean space $\mathbb{R}^n$
- If $K_0 = -1$, ... Hyperbolic space $\mathbb{H}^n$

(if $K_0 < 0$, univ. cover is a scaling of hyp. space) to see this, we observe; for $K_0=-1$, say;

$$\begin{tikzcd} & T_m \mathbb{H}^n \arrow[r] & \mathbb{H}^n \arrow[d] \\ \mathbb{R}^n \arrow[ur] \arrow[dr] \arrow[urr, bend left, "F"] \arrow[drr, bend right, "G"'] & & \\ & T_pM \arrow[r] & M \end{tikzcd}$$

$F^*(g_{\mathbb{H}^n}) = G^*(g)$, F, G loc. isom.

F is a diffeo, G the univ. covering

$\Rightarrow$ $G\circ F^{-1}: \mathbb{H}^n \to M$

locally isom. univ. covering. □

<u>So</u> $M \underset{\text{isom}}{\cong} \mathbb{H}^n/\Gamma$, $\pi_1 M \cong \Gamma < \mathrm{Isom}(\mathbb{H}^n)$. Similarly for $\mathbb{R}^n$

[see first homework] the covering transformations are isometries.

<u>theorem</u> If (M,g) is complete w/ constant curvature 1, then the universal cover is S^n.

pf. Suppose (M,g) has const. curv. $K_0>0$. Using Homework again

$F: B_{\pi/\sqrt{K_0}}(0) \longrightarrow M$ local diffeo and

$\mathbb{R}^n$ $\quad F^*(g) = d\rho^2 + \dfrac{\sin^2(\rho\sqrt{K_0})}{K_0}\, g_S$

Observe that as $F^*(g)\big|_{S_r(0)}$ has diameter $\to 0$ as $r \nearrow \pi/\sqrt{K_0}$.

$\Rightarrow$ F extends to a homeo. from one pt compactification of $B_{\pi/\sqrt{K_0}}(0)$

$\widehat{B_{\pi/\sqrt{K_0}}(0)}$, which is $\overset{\text{homeo}}{\cong} S^n$. We have

$$\begin{tikzcd} \widehat{B_{\pi/\sqrt{K_0}}(0)} \arrow[r, "F", "\cong"'] \arrow[dr, "\text{local homeo}"'] \arrow[dr, bend right=60, "G"'] & S^n_{1/\sqrt{K_0}} \arrow[d] \\ & M \end{tikzcd}$$

isom on $B_{\pi/\sqrt{K_0}}(0)$, $\widehat{B_{\pi/\sqrt{K_0}}(0)}$ is also <u>completion</u> of $B_{\pi/\sqrt{K_0}}(0)$, so F is an isometry

local isom on $B_{\pi/\sqrt{K_0}}(0)$ ... $\Rightarrow$ $G\circ F^{-1}: S^n \to M$ is a loc. isom. univ. covering. □

So $M \cong S^n/\Gamma$, $\Gamma < \mathrm{Isom}(S^n)$

$M = \tilde{M}/\Gamma$, w/ $\tilde{M} = \mathbb{H}^n, \mathbb{R}^n, S^n$ — space forms, ... more later

Theorem (Bonnet, Myers) (M,g) complete Riem mfld. Suppose

$$Ric_M(v) \geq \frac{1}{r^2} > 0$$

$\forall$ $m \in M$, $v \in T_mM$. Then M is cpct w/ diameter $\leq \pi r$.

Corollary (M,g) complete, Ric as above. Then $|\pi_1 M| < \infty$.

proof Let $p: \tilde{M} \to M$ be the univ. covering. $p^*(g)$ satisfies same curvature hyp. as g, so $\tilde{M}$ is cpct. Covering gp acts prop. discont., so $\tilde{M}$ is compact. $\square$

$\boxed{Ric_M}$ is avg of sect. curv., so same thm is true if $K(\sigma) \geq \frac{1}{r^2}$.

Proof of thm: Let $\gamma: [0,1] \to M$ be a geodesic of length $> \pi r$. We will show that γ is not minimizing. Since any two points are connected by a minimizing geodesic, such a geodesic must have length $\leq \pi r$, and will be done.

This is similar to before, we sketch the argument.

Let $\xi_1^0, \dots, \xi_n^0$ be parallel o.n. frame fields over γ w/ $\gamma'(t) = l(\gamma)\xi_1^0(t)$

$\forall$ $j = 2, \dots, n$ set

$$\xi_j(t) = \sin(\pi t)\xi_j^0(t)$$

As before, the 2^{nd} variation of energy for γ by ξ_j is

$$E_j''(0) = 2\int_0^1 \sin^2(\pi t)\left(\pi^2 - l(\gamma)^2 K(\xi_1, \xi_j)(t)\right) dt$$

$$\Rightarrow \sum_{j=2}^n E_j''(0) = 2\int_0^1 \sin^2(\pi t)(n-1)\left(\pi^2 - l(\gamma)^2 Ric_{\gamma(t)}(\xi_1(t))\right) dt$$

$$< 2\int_0^1 \sin^2(\pi t)(n-1)\left(\pi^2 - \pi^2 r^2 \underbrace{Ric_{\gamma(t)}(\xi_1(t))}_{\geq \frac{1}{r^2}}\right) dt \leq 0$$

$E_j''(0) < 0$ for some j, as before, it follows that γ is not locally minimizing $\square$. hence not minimizing.

Under the assumption of positive sectional curvature, one has:

<u>Theorem</u> (Synge, Weinstein) If (M^n, g) is complete Riemannian, w/ positive sectional curvature $K(\sigma) \geq \frac{1}{r^2} > 0$, then

① If n is even and M is orientable, then $\pi_1 M = \{1\}$.

② If n is odd, then M is orientable.

<u>pf</u> see DC 203 — 207

<u>Ex</u> $\mathbb{RP}^n = \mathbb{S}^n/_{\{\pm I\}}$, $\{\pm I\} < O_n = \mathrm{Isom}(\mathbb{S}^n)$.

<u>Ex</u> In odd dimension:

$$\mathbb{S}^{2n+1} = \{ z \in \mathbb{C}^{n+1} \mid |z| = 1 \}$$

Let $p_1, \dots, p_n, q \in \mathbb{Z}_{>0}$, $\gcd(p_i, q) = 1$ $\forall i$. then consider. $\mathbb{Z}/q\mathbb{Z}$ acting on $\mathbb{S}^{2n+1}$:

$$\mathbb{Z}/q\mathbb{Z} \times \mathbb{S}^{2n+1} \longrightarrow \mathbb{S}^{2n+1} \qquad \Rightarrow \qquad \mathbb{Z}/q\mathbb{Z} \hookrightarrow U_{n+1} < SO_{2n+2} = \mathrm{Isom}^+(\mathbb{S}^{2n+1}).$$

$$(k, (z_0, \dots, z_n)) \longmapsto (e^{\frac{2\pi i k}{q}} z_0, e^{\frac{2\pi i k p_1}{q}} z_1, \dots, e^{\frac{2\pi i k p_n}{q}} z_n)$$

$$L(p_1, \dots, p_n, q) = \mathbb{S}^{2n+1}/(\mathbb{Z}/q\mathbb{Z}) \qquad \underline{\text{Lens Space}}. \qquad \pi_1(L(p_1, \dots, p_n, q)) \cong \mathbb{Z}/q\mathbb{Z}.$$

These are examples of space forms: A Riem. mfd of constant curvature is called a <u>space form</u> and as we have already noted, is a quotient of $\tilde{M} = \mathbb{H}^n, \mathbb{R}^n$, or $\mathbb{S}^n$ (after normalizing curvature $-1, 0, 1$) by a subgroup $\Gamma < \mathrm{Isom}\,\tilde{M}$, acting properly discontinuous and freely.

<u>Proposition A</u>: Let (M, g) be a connected, homogeneous Riemannian mfd. $\Gamma < \mathrm{Isom}(M, g)$ acts properly discontinuous on M if and only if Γ is discrete.

(p92-96)

To prove this, we will need the following, which is of independent interest.

If G acts on M and N, then a map $f: M \to N$ is equivariant w.r.t. these actions if $\forall g \in G, m \in M$, we have

$$g \cdot (f(m)) = f(g \cdot m).$$

Insert into $\boxed{\otimes}$ **Lecture 21** (92)

M complete, const curvature (say $0, 1, -1$) equivalently,

Space forms: $M = \tilde{M}/\Gamma$, $\tilde{M} = \mathbb{H}^n, \mathbb{S}^n, \mathbb{R}^n$, $\Gamma < \mathrm{Isom}\,\tilde{M}$, p.d. on $\tilde{M}$ & free

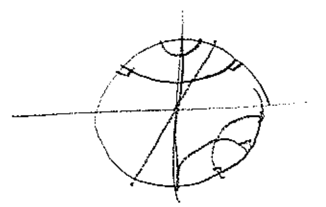
Last time we discussed some examples for $\mathbb{S}^n$ & $\mathbb{R}^n$.

– A general construction shortly, but first, some concrete examples.

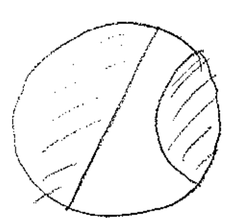
Ex. $\mathbb{H}^2$ – several models: hyperboloid, upper ½-plane – isometric since $K = -1$.

also have unit disk $\mathbb{H}^2 = \left(\{z \in \mathbb{C} \mid |z| < 1\},\ g_{\mathbb{H}} = \frac{4}{(1-|z|^2)^2} g_{\mathbb{E}}\right)$ – conformal

Geodesics are (up to parametrization) arcs of lines and circles $\perp$ to $\mathbb{S}^1$



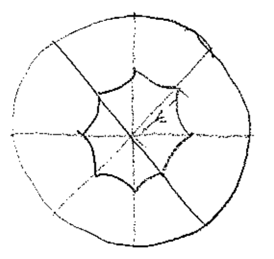
A ½-plane is a subset consisting of all points on one side of a geodesic



A polygon is the intersection ½-planes whose ∂'s are a locally finite set of geodesics.

$\forall\ n \geq 3$, consider the 1-parameter family of regular n-gons "centered" at O; $0 < t < 1$

Note: $O_2 < \mathrm{Isom}(\mathbb{H}^2)$
$\quad \| $
$\mathrm{Stab}(0)$
$\mathrm{Isom}(\mathbb{D})$

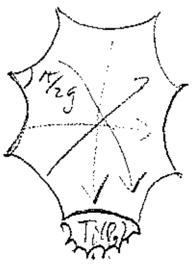


Let $\theta_n(t)$ = interior angle
clearly varies continuously w/ t.
$t \to 0$, $\theta_n(t) \to$ int. angle of regular n-gon in $\mathbb{R}^2$ – scale up $= \frac{(n-2)\pi}{n}$

as $t \to 1$, $\theta_n(t) \to 0$

$\forall g \geq 2$, By the intermediate value thm, there is some $0<t<1$ s.t.

$$0 < \theta_{4g}(t) = \frac{2\pi}{4g} = \frac{\pi}{2g} < \frac{(4g-2)\pi}{4g}$$. Let P_g be this polygon

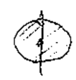
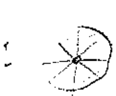


label the sides of P_g $\gamma_1, \dots, \gamma_{4g}$, then consider the unique isometries $T_1, \dots, T_{2g}$ where $T_i(\gamma_i) = \gamma_{i+2g}$ and $T_i(\mathring{P}_g) \cap \mathring{P}_g = \emptyset$ — such T_i exists b/c $\mathrm{Isom}(\mathbb{H}^2)$ acts transitively on $\mathbb{H}^2$ w/ $\mathrm{Stab}(z) \cong O_2$ $\forall z \in \mathbb{H}^2$

Consider the surface obtained by identifying opposite sides via $\{T_i\}$;

$S = P_g/\sim$ where $x \sim T_i(x)$ $\forall x \in \gamma_i$ and all vertices are equivalent.

there is a metric on S making it locally isometric to $\mathbb{H}^2$:

$x \in \mathrm{int}\, P_g$:  , $x \in \mathrm{int}(\gamma_i)$:  x = vertex : 

these local isometries (a) give coords to $\mathbb{H}^2$ exhibits S as a smooth surface
(b) prove that the metric is induced by a Riem metric w/ constant curvature $K=-1$.

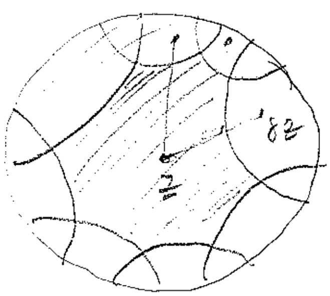
$\therefore$ $S = \mathbb{H}^2/\Gamma$ In fact, $\Gamma = \langle T_1, \dots, T_{2g} \rangle < \mathrm{Isom}(\mathbb{H}^2)$

$\therefore$ Every closed surface of genus $g \geq 2$ admits a <u>hyperbolic structure</u> — a metric of curvature $=-1$.

<u>Theorem</u>: If $S = \mathbb{H}^2/\Gamma$ is a hyperbolic surface of genus $g\geq 2$, then $\mathrm{area}(S) = 2\pi|\chi(S)| = (4g-4)\pi$.

<u>proof</u> Let $z \in \mathbb{H}^2$, consider

$$D_\Gamma(z) = \{ w \in \mathbb{H}^2 \mid d(z,w) \leq d(z, \gamma w) \ \forall \gamma \in \Gamma \}$$
$$= \{ w \in \mathbb{H}^2 \mid d(z,w) \leq d(\gamma z, w) \ \forall \gamma \in \Gamma \}$$

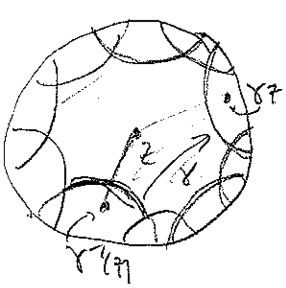


Action prop. disc $\Rightarrow$ $\Gamma\cdot z$ is discrete,

$$D_\Gamma(z) = \bigcap_{\gamma\in\Gamma} H(z,\gamma z)$$

$$H(z,\gamma z) = \{w \in \mathbb{H}^2 \mid d(z,w) \le d(\gamma z, w)\} = \tfrac{1}{2}\text{-plane}$$

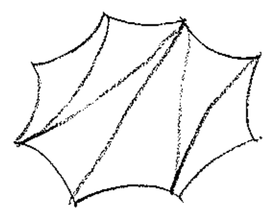
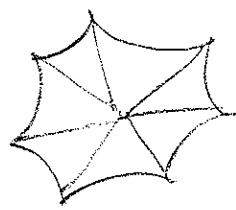
$D_\Gamma(z)$ is a cpct polygon and $S = D_\Gamma(z)/\sim$:



- Every orbit has $\underbrace{\text{a rep. in } D_\Gamma(z)}_{\text{a closest pt.}}$, interior pts have exactly one
- $d_S(p(z),p(w)) = d(z,w)$ $\forall\, w \in D_\Gamma(z)$ since o.w. $\exists$ a closer pt to z in $\Gamma\cdot w$ $\Downarrow\uparrow$.

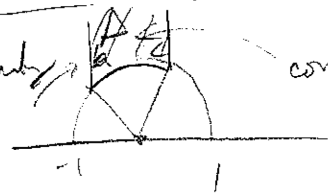
$\therefore$ $D_\Gamma(z) \subset \mathrm{diam}(S)$-ball about z.

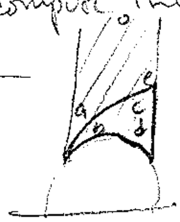
Now, subdivide $D_\Gamma(z)$ into triangles

 or 

Exercise: $\mathrm{Area}(\Delta(a,b,c)) = \pi - (a+b+c)$ where a,b,c = interior angles

idea: assume $a = 0$ "vertex at ∞" in upper ½-plane 

apply isom  compute the integral $\int_\Delta \frac{dx\,dy}{y^2} = \pi - (b+c)$

In general: 

$$\pi - (a+b+d) = \pi - (a+c+d) + \pi - (a+e)$$

"triangulation" of S w $V = \#$ vert, $E = \#$ edge, $F = \#$ faces. $3F = 2E$

$$2g-2 = -V+E-F$$
$$4g-4 = -2V+2E-2F$$
$$= F-2V$$

$$\mathrm{Area}(S) = \sum_{i=1}^{F} \mathrm{Area}(\Delta_i)$$ 

$$= \sum_{i=1}^{F}\left(\pi - (a_i^1+a_i^2+a_i^3)\right) = F\pi - \sum_{i=1}^{F}(a_i^1+a_i^2+a_i^3)$$

$$= F\pi - 2\pi V = \pi(F-2V) = (4g-4)\pi \quad \square$$

Dimension of the space of hyperbolic structures on a surface:

We can get a rough estimate on this as follows:

Write $S = \mathbb{H}^2/\Gamma$ and let $\rho_0 : \pi_1 S \to \Gamma$ be the associated isomorphism.

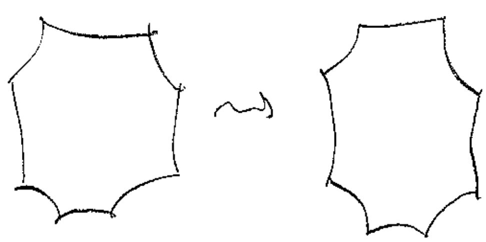
Then $\rho_0 \in \mathrm{Hom}(\pi_1(S), \mathrm{Isom}\,\mathbb{H}^2)$ w/ image acting prop. disc & freely, $\underset{1\text{-}1}{\wedge}$

$$\mathrm{Hom}(\pi_1(S); \mathrm{Isom}\,\mathbb{H}^2) \cong \left\{ (X_1, Y_1, \ldots, X_g, Y_g) \in (\mathrm{Isom}\,\mathbb{H}^2)^{2g} \;\middle|\; \prod_{i=1}^{g} [X_i, Y_i] = 1 \right\}$$

$$\rho \longmapsto (\rho(a_1), \rho(b_1), \ldots, \rho(a_g), \rho(b_g))$$

where $\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$

$\exists$ nbhd $\mathcal{U}$ of ρ_0 s.t. $\forall \rho \in \mathcal{U}$, $\rho(\pi_1(S))$ also acts prop. disc & freely



Fund. domain for ρ close to Fund domain for ρ_0. . . .

If ρ & ρ' are conjugate, then $\mathbb{H}^2/\rho(\pi_1(S))$ isometric to $\mathbb{H}^2/\rho'(\pi_1(S))$

So, a nbhd of ρ_0 in

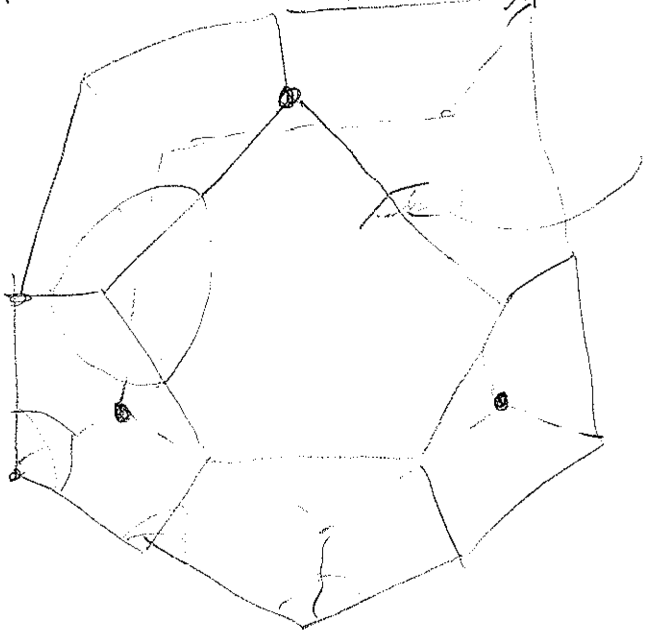
$$\mathrm{Hom}(\pi_1(S); \mathrm{Isom}\,\mathbb{H}^2)/\text{conj}.$$

parametrizes deformations of hyperbolic metric on S

$$\dim = \underbrace{3 \cdot 2g}_{(\mathrm{Isom}\,\mathbb{H}^2)^{2g}} - \underbrace{3}_{\Pi[X_i, Y_i]} - \underbrace{3}_{\text{conj.}} = 6g - 6$$

$\mathbb{H}^3$ We can also construct 3-mfd $\mathbb{H}^3/\Gamma$ by similar procedure of gluing 3-dim'l hyperbolic polyhedra:

EX $\exists$ 1-parameter family of hyperbolic regular dodecahedra: D_t



dihedral angles $\theta(t)$ vary continuously for $0<t<1$

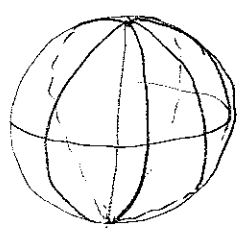
and $\lim_{t\to 0} \theta(t) =$ Euc. angle $\approx 116.6°$.

and $\lim_{t\to\infty} \theta(t) = 60°$.

for some t, get $\theta(t) = 72°$

identify opposite sides by hyp. isometry that twists "3/10's of the way around" for all pairs of sides

each edge is identified w/ exactly 4 others, so get nbhds of 5 edges gluing together



obviously o.k. for points on int D_t and on faces. Can check it for vertex as well. - link has regular icosahedral tesselation.