Lecture 1

Riemannian geometry (+ general differential geometry)

$M = M^n$ smooth n-manifold (Hausdorff, 2^{nd} countable, $\partial M = \emptyset$ for simplicity — $\partial M \neq \emptyset$ then $\exists N$, $\partial N = \emptyset$, $f: N \to \mathbb{R}$, 0 regular value s.t. $M = f^{-1}([0,\infty))$ ...)

Defn A Riemannian metric on M is a smoothly varying inner product on the tangent spaces of M.

That is, a section

$$g \in \Gamma(T^*M \otimes T^*M) = \Gamma(\otimes^2 T^*M) = \{\sigma : M \to \otimes^2 T^*M \mid \pi \circ \sigma = id_M\}$$

$\pi : \otimes^2 T^*M \to M$ can. projection

s.t. writing $g_m \in \otimes^2 T^*_m M$, $m \in M$ we have

① $g_m(v, w) = g_m(w, v)$ $\forall\, v, w \in T_m M$ (symmetry)

② $g_m(v, v) \geq 0$ $\forall\, v \in T_m M$ and $g_m(v,v) = 0 \iff v = 0$ (positive definite)

Sometimes write $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$

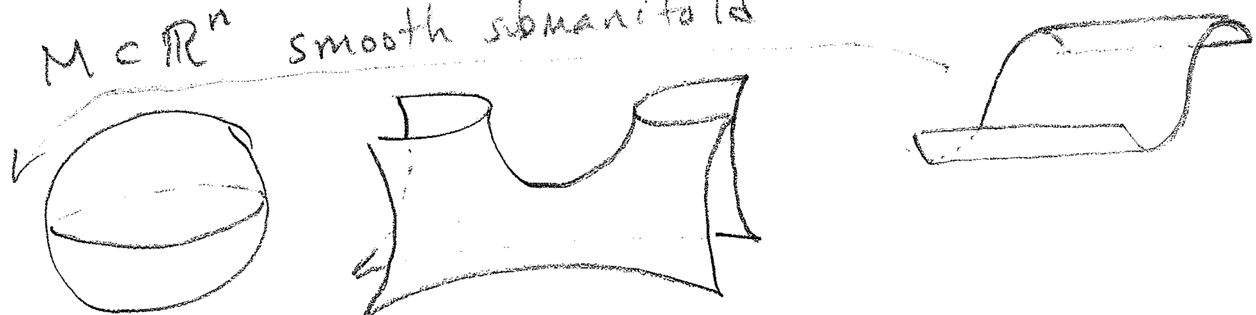
$x = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$ local coordinates, $\{\frac{\partial}{\partial x_i}\}_{i=1}^n \subset \mathfrak{X}(U) = \Gamma(TU)$ C^∞ vector fields

$\Rightarrow g_{ij} \in C^\infty(U)$, given by $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ $i, j \in \{1, \ldots, n\}$

Ex $\mathbb{R}^n$, g_m = standard inner product on $T_m\mathbb{R}^n = \mathbb{R}^n$, $\forall m$

$g_{ij} = \delta_{ij}$. Write $(\mathbb{R}^n, g_{std})$ = Euclidean space of dimension n

Ex $M \subset \mathbb{R}^n$ smooth submanifold



$T_mM \subset T_m\mathbb{R}^n = \mathbb{R}^n$ g_m = restriction of std inner product

Defn A smooth manifold together w/ a Riemannian metric (M,g) is called a Riemannian manifold.

Theorem. Every M admits a Riemannian metric proof - later

A Riemannian manifold admits a canonical distance function (a metric in the traditional sense):

First, given a piecewise smooth path

$\gamma : [a,b] \longrightarrow M$

define its length

$$\ell(\gamma) = \ell_g(\gamma) = \int_a^b |\gamma'(t)|\, dt \quad \text{where } |v| = \sqrt{g_m(v,v)} \ \forall m \in M,\ v \in T_mM.$$

set

$$d(m_1,m_2) = d_g(m_1,m_2) = \inf\{\ell(\gamma) \mid \gamma : [a,b] \to M \text{ p.w. smooth}, \gamma(a) = m_1, \gamma(b) = m_2\}.$$

Theorem. Given (M,g) Riem. mfd., d_g is a metric.

Proof - later

The natural equivalence relation comes from

__Def'n__ Given Riemannian mfds (M, g), (M', g'), an __isometry__ is a diffeomorphism $f: M \longrightarrow M'$ s.t.

$$g'_{f(m)}(df_m(v), df_m(w)) = g_m(v, w) \quad \forall\ m \in M,\ v, w \in T_m M.$$

Clearly, if $f: M \to M'$ is an isometry, then

$$d_{g'}(f(m_1), f(m_2)) = d_g(m_1, m_2).$$

Converse also true (Palais, PAMS '57) — d_g determines smooth structure [Maybe return to this later]

__Course Outline__ I. Basic structure of a Riemannian manifold: metric and geodesics — locally length minimizing paths. Key tool: covariant derivative/connection. DoCarmo Ch. 1-3.

[metric/geodesics — intuition | covariant derivative — formalism]

II. __THE__ invariant of a Riemannian metric: Curvature measures the deviation from being locally isometric to $(\mathbb{R}^n, g_{std})$

geometrically: How fast geodesics diverge (infinitesimally):

formally: "twistedness of covariant deriv"

like $\mathbb{R}^2$ 0 slower >0 faster <0

DC Ch. 4-6, Spivak 6, 7, 8

III Local to global: Effects of positive, nonpositive, negative curvature on global topology of a Riem. mfd. Bonnet-Myers Thm, Cartan-Hadamard Thm, Comparison geometry and Flat Torus thm. Constant curvature manifolds and space forms. DC Ch. 7-9, Bridson-Haefliger Ch. II1, II6, II7.

[* Discuss Syllabus *]

__Theorem__ Every smooth mfd M admits a Riemannian metric.

__Lemma__: V real vector space, g_0, g_1 inner products, $t \in [0,1]$, then

$$g_t = tg_1 + (1-t)g_0 \qquad \Big(g_t(u,v) = t(g_1(u,v)) + (1-t)(g_0(u,v)) \Big)$$

is an inner product.

__Proof__ $\otimes^2 V^*$ is a vector space, so $g_t \in \otimes^2 V^* \; \forall t$.

In fact $Sym^2 V^*$ = symmetric 2-tensors is a subspace so $g_t \in Sym^2 V^*$. (easy check) Given $v \in V$

$g_t(v,v) = tg_1(v,v) + (1-t)g_0(v,v) \geq 0$ since all terms ≥ 0.

if $= 0$, then all terms $= 0$ $\Rightarrow$ $g_1(v,v) = 0$ or $g_0(v,v) = 0$ $\Rightarrow v = 0$. $\square$

__Corollary__ Space of Riemannian metrics on M is an (open) cone in $\Gamma(\otimes^2 T^*M)$

Proof of Thm: $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha\in J}$ an atlas of charts on M, $\{\rho_j : M \to \mathbb{R}\}_{j=1}^{\infty}$ a loc. finite part. of unity subordinate to the atlas — say $\operatorname{supp}(\rho_j) \subset U_{\alpha(j)}$.

Define $g \in \Gamma(\otimes^2 T^*M)$ by

$$g = \sum_{j=1}^{\infty} \rho_j \cdot \phi_{\alpha(j)}^*(g_{std}) \qquad (\text{locally finite sum})$$

Here $\phi^*(g_{std})_m(u,v) = g_{std}(d\phi_m(u), d\phi_m(v))$

By Lemma, g sym., pos. def — Riem. metric. $\square$

Alternatively: Whitney Embedding thm $\Rightarrow f: M \hookrightarrow \mathbb{R}^{2n+1}$ smooth embedding. Restrict metric: $f^*(g_{std})$.

Def'n More generally, if $f: M \to N$ is an immersion of Riemannian manifolds (M,g) to (N,h) s.t. $f^*(h) = g$, then f is called an isometric immersion. If f is a local diffeomorphism and an isometric immersion, it is called a local isometry. [isometry is special case]

Nash Embedding Theorem: Every (M,g) admits an isometric embedding into some Euclidean space.

(see Nash, Ann. of Math '54)

Lecture 2

The distance function

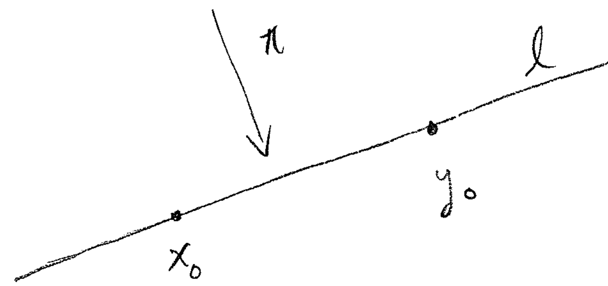
EX $(\mathbb{R}^n, g_{std})$. The distance between $x_0, y_0 \in \mathbb{R}^n$?

Consider the line

$$l = \{ t x_0 + (1-t) y_0 \mid t \in \mathbb{R} \}$$

and the $\perp$ projection

$$\pi : \mathbb{R}^n \longrightarrow l.$$



Observe (calculation):

$$\forall\, v \in T_m \mathbb{R}^n = \mathbb{R}^n, \quad |d\pi_m(v)| \le |v| \quad \text{w/ equality} \Leftrightarrow v \parallel l.$$

$\Rightarrow \forall\, \gamma : [a,b] \longrightarrow \mathbb{R}^n$

$$l(\gamma) \ge l(\pi \gamma) \quad \text{with equality} \Leftrightarrow \gamma'(t) \parallel l \quad \forall\, t \in [a,b].$$

$\Rightarrow$ straight segment minimizes length. Moreover, if γ is any length minimizer,

then $\dfrac{\gamma'(t)}{|\gamma'(t)|} = \pm \dfrac{y_0 - x_0}{|y_0 - x_0|}$ i.e. γ is made up of arcs of flow lines for the constant vector field $\dfrac{y_0 - x_0}{|y_0 - x_0|}$.

$\therefore$ length min. path is $!$ straight line segment from x_0 to y_0, and

$$d_{g_{std}} = d_{std} = \text{the standard Euclidean distance function.}$$

Theorem (M, g) connected Riemannian manifold, $d_g = d$ is a metric.

<u>proof</u> $d(m,p)=d(p,m)$, $d(p,q)\le d(p,m)+d(m,q)$, $d(m,m)=0$

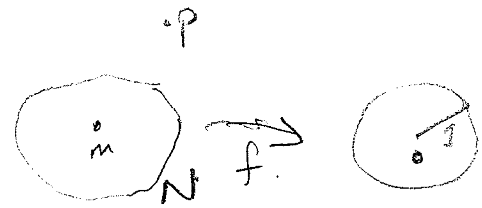
— obvious —

<u>$d(m,p)>0$ if $m\neq p$</u> :

Let $f: N \to \bar{B}(0,1)\subset \mathbb{R}^n$ be a diffeomorphism of a cpct nbhd N of m not containing p

onto the ball of radius 1

in $(\mathbb{R}^n, g_{std})$



(W.L.O.G can assume $f=F|_N$, $F: W\to B(0,2)$ is a diffeo.)

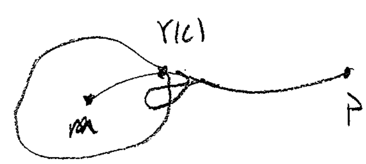
so that $\exists$ $K>1$ s.t.

$$\frac{1}{K}\le\frac{|df_m(v)|}{|v|}\le K \qquad \forall m\in N \;\; v\in T_mM-\{0\}$$

For any $\gamma:[a,b]\to M$, $\gamma(a)=m$, $\gamma(b)=p$ $\exists$ $c\in(a,b)$ st.

$f(\gamma(c))\in\partial N=f^{-1}(\partial\bar{B}(0,1))$

$f(\gamma[a,c])\subset N$.



Then $\ell(\gamma)\ge\ell(\gamma|_{[a,c]})\ge\frac{1}{K}\,\ell(f(\gamma)|_{[a,c]})\ge\frac{1}{K}\cdot 1$

So, $d(m,p)\ge\frac{1}{K}>0$. $\square$

<u>Ex</u> $(\mathbb{R}^n, g_{std})$

<u>Ex</u> $(M, g_{std}|_M)$ if $M \subset \mathbb{R}^d$

<u>unit n-Sphere</u>:

$$S^n = \{ m \in \mathbb{R}^{n+1} \mid |m| = 1 \} = f^{-1}(1) \text{ w/ } f(x) = \sum_{i=1}^{n+1} x_i^2 .$$

<u>Ex</u> $(M,g), (N,h)$ Riemannian mfds. — <u>product metric</u>

$(M \times N, g \times h)$: $\pi : M \times N \to M$, $p : M \times N \to N$

$g \times h = \pi^*(g) + p^*(h)$.

or identifying $T_{(m,q)}(M \times N) = T_m M \times T_q N$, we have

$g \times h((u,v),(w,s)) = g(u,w) + h(v,s)$

<u>Flat n-torus</u>: $T^n = S^1 \times .. \times S^1$

<u>Ex</u> $f : M \to N$ loc. diffeomorphism, g Riem. metric on N, then $f^*(g)$ is (the) Riemannian metric on M making f into a local isometry

<u>Ex</u> Suppose $(\tilde{M}, \tilde{g})$ is a Riem. mfd., Γ a group and

$\Gamma \times \tilde{M} \to \tilde{M}$

a proper discontinuous, free action:

$\Gamma \times \tilde{M} \longrightarrow \tilde{M}$ smooth

$(\gamma, \tilde{m}) \mapsto \gamma . m$

satisfies

$(\gamma_1 \gamma_2) \cdot \tilde{m} = \gamma_1 (\gamma_2 \cdot \tilde{m}) \quad \forall \gamma_1, \gamma_2 \in \Gamma, \tilde{m} \in \tilde{M}$ } action

$e \cdot \tilde{m} = \tilde{m} \quad \forall \tilde{m} \in \tilde{M}$, e = identity in Γ.

$|\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}| < \infty \quad \forall$ cpct $K \subset \tilde{M}$ } prop. disc.

$\gamma \tilde{m} = \tilde{m} \Rightarrow \gamma = e$ } free.

then $M = \tilde{M}/\Gamma$ is a smooth mfd. and $f : \tilde{M} \to M$ obvious projection is a local diffeo.

If Γ acts by isometries :

$m \mapsto \gamma \cdot m$ an isometry $\forall \gamma \in \Gamma$

then $\exists !$ Riem. metric g on M s.t. $f^*(g) = \tilde{g}$.

(see exercises)

Detour Defn' : A Lie group is both a group and a smooth mfd in which the two structures are compatible, that is the two maps

$m : G \times G \longrightarrow G$

$i : G \longrightarrow G$

given by $m(g,h) = g \cdot h$ and $i(g) = g^{-1}$, are required to be smooth.

EX $(\mathbb{R}^n, +)$ - obvious

EX $(\mathbb{T}^n, +)$ - $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ $2\pi\mathbb{Z}^n \triangleleft \mathbb{R}^n$

(see homework)

EX $\underset{\substack{\uparrow \\ \det \neq 0}}{GL_n\mathbb{R}} \underset{\text{opn}}{\subset} M_n\mathbb{R} =$ $n\times n$ matrices.

matrix multiplication — entries are polynomials
matrix inversion — entries are rational functions
$\left(\frac{1}{\det A}\operatorname{adj} A = A^{-1},\ \operatorname{adj} A = \text{adjoint matrix of cofactors}\right)$

similarly, $GL_n\mathbb{C}$.

EX $\underset{\substack{\uparrow \\ \det = 1}}{SL_n\mathbb{R}, SL_n\mathbb{C}}$. — det is a smooth map and a homomorphism.

by Sard's Thrm, a.e. $a \in \mathbb{R}$ or $\mathbb{C}$ is a regular value

Pick one $a \neq 0$ and let $A \in SL_n\mathbb{K}$ be st. $\det A = a^{-1}$.

then $\det^{-1}(1) = A \cdot \det^{-1}(a)$ is a smooth submanifold.

restriction of smooth is smooth, so $SL_n\mathbb{R}, SL_n\mathbb{C}$ are Lie subgroups.

$p, q \in \mathbb{Z}_{\geq 0}$ $\quad p+q=n$

define $O(p,q) < GL_n\mathbb{R}$ by

$$O(p,q) = \{A \in GL_n\mathbb{R} \mid A^T J_{p,q} A = J_{p,q}\}$$

here $J_{p,q} = \begin{pmatrix} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{p} & 0 \\ 0 & \underbrace{\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}}_{q} \end{pmatrix}$ $\quad J_{p,q}$ represents a symmetric bilinear form

on $\mathbb{R}^n$ by

$$B_{p,q}(u,v) = u^T J_{p,q} v$$

Standard symmetric bilinear form of signature p,q.

$O(p,q)$ is precisely the set of linear automorphisms preserving the form

Set $J = J_{p,q}$. Then

$$A \in O(p,q) \iff A^T J A = J \iff (A^T J A)_{ij} = J_{ij} \quad \forall 1 \le i,j \le n$$
$$\iff e_i^T A^T J A e_j = e_i^T J e_j \quad \forall 1 \le i,j \le n$$
$$\iff B_{p,q}(Ae_i, Ae_j) = B_{p,q}(e_i, e_j) \quad \forall 1 \le i,j \le n$$
$$\iff B_{p,q}(Au, Av) = B_{p,q}(u,v) \quad \forall \ u,v \in \mathbb{R}^n$$

To see $O(p,q)$ is a Lie subgroup, follow approach similar to $O_n = O(n,0)$:

Define $F: M_n\mathbb{R} \to Sym_n\mathbb{R}$ = v.s. of symmetric $n \times n$ real matrices

by $F(A) = A^T J A$

so that $F^{-1}(J) = O(p,q)$. Check that J is a regular value:

Given $A \in O(p,q)$, $B \in T_A M_n\mathbb{R}$, compute

$$dF_A(B) = \lim_{t \to 0} \frac{F(A+tB) - F(A)}{t} = A^T J B + B^T J A.$$

then any $C \in Sym_n\mathbb{R}$ can be written as $C = \frac{1}{2}C + \frac{1}{2}C^T$ and thus for $B = \frac{1}{2} J (A^{-1})^T C$ we have

$$dF_A(B) = C \qquad \text{(check this)}. \quad \square$$

Special case — $q=0 \Rightarrow O_n = O(n,0) = O(n)$.

Lecture 3

Def'n: A Lie algebra is a vectorspace V w/ a bilinear form $[\,,\,]: V \times V \to V$ s.t.

1. $[u,v] = -[v,u]$
2. $[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$ (Jacobi identity)

Ex: ⓪ V, $[,]\equiv 0$. — abelian
① $\mathfrak{X}(M)$, $[,]$ = Lie bracket.
② $M_n\mathbb{R}$, $[A,B]=AB-BA$. (exercise)

G a Lie group. A v.f. $\xi\in\mathfrak{X}(G)$ is left invariant if $d(l_g)(\xi)=\xi$ where
$l_g: G\to G$ is left multiplication:

$l_g(h)=gh$

that is, ξ must be l_g-related to itself. Value at $g\in G$ is determined by value at e = identity of G:

$$\xi_g = d(l_g)_e(\xi_e)$$

In fact, this defines an isomorphism

$$T_eG \longrightarrow \mathfrak{g}$$

where $\mathfrak{g}$ denotes the vector space of left invariant vector fields,
(need to check that $(g\mapsto d(l_g)_e(\xi_e))$ is C^∞ — see Warner, p85)

Thm: $\mathfrak{g}$ w/ $[,]$ is a Lie algebra.
this is the Lie algebra of the Lie group G.
Proof: Need to check, $\xi,\eta\in\mathfrak{g}$, then $[\xi,\eta]\in\mathfrak{g}$.
But ξ,η l_g-related to themselves $\Rightarrow$ $[\xi,\eta]$ is l_g-related to itself. $\square$

Ex ① Lie algebra of $\mathbb{R}^n$ is $\mathbb{R}^n$ w/ $[\,,\,] \equiv 0$.

② Lie algebra of $GL_n\mathbb{K}$ is $\mathfrak{gl}_n\mathbb{K} = M_n\mathbb{K}$ w/ $[A,B] = AB - BA$ (exercise)
$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

③ Lie algebra of $O(p,q)$ is

$$\mathfrak{o}(p,q) = \{ A \in M_n\mathbb{R} \mid J_{p,q}A + A^T J_{pq} = 0 \} = \ker dF_I \quad , F(B) = B^T J B.$$

[We'll come back to $SL_n\mathbb{K}$]

Homomorphisms:

Def'n A <u>homomorphism of Lie groups</u> is a smooth map

$$\phi : G \longrightarrow H$$

which is also a homomorphism of groups: $\phi(gg') = \phi(g)\phi(g')$ $\forall g, g' \in G$.

A <u>homomorphism of Lie algebras</u> is a linear map

$$\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$$

st $\psi([\xi,\eta]) = [\psi\xi, \psi\eta]$.

If $\phi : G \longrightarrow H$ is a homomorphism of Lie groups, then $\phi(e) = e'$ ($e \in G$, $e' \in H$ identity elements) So $d\phi_e : T_eG \longrightarrow T_{e'}G$.

identifying $T_eG = \mathfrak{g}$ and $T_{e'}H = \mathfrak{h}$, $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H respectively, we can write $d\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ for the map making

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\phi"] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathfrak{h} \arrow[d] \\ T_eG \arrow[r, "d\phi_e"] & T_{e'}H \end{tikzcd}$$

commute.

Theorem Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and $\phi: G \to H$ a homomorphism. Then

1. ξ and $d\phi(\xi)$ are ϕ-related $\forall\, \xi \in \mathfrak{g}$, and
2. $d\phi: \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

proof Let $\tilde{\xi} = d\phi(\xi) \in \mathfrak{h}$. Since ϕ is a homomorphism, $\forall\, g \in G$ we have

$$\phi \circ l_g = l_{\phi(g)} \circ \phi .$$

so

$$\begin{aligned} \tilde{\xi}_{\phi(g)} &= dl_{\phi(g)}|_{e'}(\tilde{\xi}_{e'}) = d(l_{\phi(g)})_{e'}(d\phi_e(\xi_e)) \\ &= d(l_{\phi(g)} \circ \phi)_e(\xi_e) = d(\phi \circ l_g)_e(\xi_e) \\ &= d\phi_g(dl_g)_e \xi_e) = d\phi_g(\xi_g). \end{aligned}$$

so, ξ and $\tilde{\xi} = d\phi(\xi)$ are ϕ-related.

It follows that $\forall\, \xi, \eta \in \mathfrak{g}$,

$$[\xi, \eta] \text{ is } \phi\text{-related to } [d\phi(\xi), d\phi(\eta)]$$

$$\Rightarrow \quad d\phi_e([\xi,\eta]_e) = [d\phi(\xi), d\phi(\eta)]_{e'}$$

By the definition of $d\phi$, it follows that $d\phi([\xi,\eta]) = [d\phi(\xi), d\phi(\eta)]$

(! elt of $\mathfrak{h}$ $= d\phi_e([\xi,\eta]_e)$ at e') $\square$

In fact, $d\phi$ determines ϕ, if G is connected:

Theorem Suppose $G, H, \mathfrak{g}, \mathfrak{h}$ as above w/ G connected. If $\phi, \psi: G \to H$ are homomorphisms w/ $d\phi = d\psi$, then $\phi = \psi$.

Proof See Warner p. 92.