Lecture 22 (pick back up from $\boxed{\circledast}$ on p 91)

(pick back up from $\boxed{\circledast}$ on p 91)

Proposition B: Suppose G acts on M and N and $f: M \to N$ is a proper equivariant surjective map. Then the action on M is properly discontinuous iff the action on N is.

Proof: Given $K \subset N$ cpct, $f^{-1}(K)$ is cpct. Furthermore, given $g \in G$

$$gK \cap K \neq \emptyset \underset{f \text{ onto}}{\iff} f^{-1}(gK \cap K) \neq \emptyset \iff f^{-1}(gK) \cap f^{-1}(K) \neq \emptyset$$
$$\iff g \cdot f^{-1}(K) \cap f^{-1}(K) \neq \emptyset$$

$\therefore$ $G \curvearrowright N$ P.D. $\Rightarrow G \curvearrowright M$ P.D.

Observe that $\forall$ $K \subset M$ cpct, $f^{-1}(f(K))$ is also cpct and

$$\{g \in G \mid g(K) \cap K \neq \emptyset\} \subset \{g \in G \mid g(f^{-1}(f(K))) \cap f^{-1}(f(K)) \neq \emptyset\}$$

So, from above, get
$G \curvearrowright M$ P.D. $\Rightarrow G \curvearrowright N$ P.D. $\square$.

Consider $\mathrm{Isom}(M, g) = G$, $m_0 \in M$ and define

$$F: G \longrightarrow M$$

by $F(\sigma) = \sigma . m_0$. Clearly F is G-equivariant.

Lemma F is a proper, submersion.

Proof M homogeneous and F G-equivariant $\Rightarrow$ F is a submersion: Indeed, Let $m_1 \in M$ be a regular value which exists by Sard's thrm, $m \in M$ any point, $\sigma \in F^{-1}(m)$, $\sigma_1 \in F^{-1}(m_1)$. By equivariance and chain rule we have

$$dF_\sigma \circ d(l_{\sigma\sigma_1^{-1}})_{\sigma_1} = d(L_{\sigma\sigma_1^{-1}})_{m_1} \circ dF_{\sigma_1}$$

where $L_\tau : M \to M$ is given by $L_\tau(p) = \tau . p$. Since RHS is onto, so is LHS.

Recall $\mathrm{Stab}(m_0)$ is a closed subgroup of $O(g|_m) \cong O_n$ and is therefore compact.

Note: $F^{-1}(m) = \sigma . \mathrm{Stab}(m_0)$, where $\sigma \in F^{-1}(m)$ is any elt.
$= l_\sigma(\mathrm{Stab}(m_0))$

$\textcircled{8}$

Let $\Sigma \subset G$ be a submfd w/ $\dim\Sigma + \dim \mathrm{Stab}(m_0) = \dim(G)$ containing σ. Then $dF|_{T_\sigma\Sigma} : T_\sigma\Sigma \to T_m M$ is an $\cong$. $\therefore$ on suff. small nbhd U of σ in Σ, $F|_U : U \xrightarrow{\cong} V \subset M$ is a diffeo, wlog. $V = B_\delta(m)$ for some $\delta > 0$.

Consider the map $G : U \times \mathrm{Stab}(m_0) \to G$
$$(\tau, \rho) \longmapsto \tau\rho$$
At every pt (τ, ρ) we can check $dG_{(\tau,\rho)}$ is an $\cong$. Restricting to smaller V if necessary, we see G is a diffeo. Let $\overline{B} = F|_U^{-1}(\overline{B}_{\delta/2}(m))$ which is cpct, then
$$G = (\overline{B} \times \mathrm{Stab}(m_0)) = F^{-1}(\overline{B}_{\delta/2}(m)) \text{ is cpct.}$$
Any compact $K \subset M$ is contained in a finite union of $\overline{B}_{\delta/2}(m_i)$, so $F^{-1}(K)$ is contained in finite union of cpcts, so cpct. $\square$

We also need

<u>Lemma</u> G a Lie group, $\Gamma < G$ a subgroup. Γ action on G is prop. disc iff Γ is discrete.

<u>Proof</u>: Fix any left inv't metric on G. Then G is homogeneous, hence complete. Since cpct sets are exactly closed, bounded sets, it suffices to show: Γ is discrete iff $\forall r > 0$, $|\{g \in \Gamma \mid g\overline{B}_r(e) \cap \overline{B}_r(e) \neq \emptyset\}| < \infty$.

But, we have
$$\Gamma \text{ is discrete} \iff |\Gamma \cap \overline{B}_{2r}(e)| < \infty \quad \forall r > 0$$
$$\iff |\{g \in \Gamma \mid g\overline{B}_r(e) \cap \overline{B}_r(e) \neq \emptyset\}| < \infty \quad \forall r > 0$$
since the two sets on the right are the <u>same</u>. $\square$

Proposition A now follows from these two lemmas & proposition B

Ex discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n) = O^+(n,1)$: Arithmetic constructions. Clearly $O^+(n,1)(\mathbb{Z}) = \{A \in O^+(n,1) \mid A_{ij} \in \mathbb{Z}\ \forall i,j\}$ is discrete. Others?

Let $k \subset \mathbb{C}$ be a number field = a finite extension of $\mathbb{Q}$
[a field containg $\mathbb{Q}$ as a subfield st. as a v.s over $\mathbb{Q}$, finite dim.]

eg $k = \mathbb{Q}$, $k = \mathbb{Q}(\sqrt{d})$ $d \in \mathbb{Z}$, squarefree

$$\begin{array}{c}\mathbb{Q}(\sqrt{d}) \\ |\ 2:1 \\ \mathbb{Q}\end{array}$$

$\zeta^d = 1$, $\mathbb{Q}(\zeta)$ = cyclotomic fields.

- Let $\sigma_1, \ldots, \sigma_d : k \to \mathbb{C}$ be the set of all Galois embeddings.
  $d = [k:\mathbb{Q}]$ Assume σ_1 = identity

eg $k = \mathbb{Q}(\sqrt{d})$, then $\sigma_1(a+b\sqrt{d}) = a+b\sqrt{d}$ and
$\sigma_2(a+b\sqrt{d}) = a-b\sqrt{d}$.

$k = \mathbb{Q}(\zeta)$, $\zeta^d = 1$. $\zeta = e^{2\pi i/d}$ to determine $\sigma_j : k \to \mathbb{C}$, only need to know $\sigma_j(\zeta) = e^{2\pi i t_j/d}$, $t_j \in \mathbb{Z}$, $e^{2\pi i t_j/d}$ primitive. (t_j generates $\mathbb{Z}/d\mathbb{Z}$)
i.e. $(t_j, d) = 1$

k is totally real if $\sigma_i(k) \subset \mathbb{R}$ $\forall i$.

Recall: $\alpha_1, \ldots, \alpha_d$ is a basis for $k/\mathbb{Q}$ iff $\det(\sigma_i(\alpha_j)) \neq 0$.

$$\sum_{j=1}^d x_j \alpha_j = 0 \iff \sum_{j=1}^d x_j \sigma_i(\alpha_j) = 0 \ \forall i \iff (\sigma_i(\alpha_j))\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = 0$$

Let $R_k \subset k$ be the ring of integers — elts of k satisfying a monic integral polynomial

Proposition $\exists\, \alpha_1, \ldots, \alpha_d \in R_k$ a basis for $k/\mathbb{Q}$ st. $\mathbb{Z}[\alpha_1, \ldots, \alpha_d] = R_k$.
i.e. $R_k = \{\sum x_i \alpha_i \mid x_i \in \mathbb{Z}\}$ — $\alpha_1, \ldots, \alpha_d$ = integral basis for k

Ex $k = \mathbb{Q}(\sqrt{d})$: $R_k = \mathcal{O}_d = \mathbb{Z}[\alpha] = \{a + b\alpha \mid a, b \in \mathbb{Z}\}$

$$\alpha = \begin{cases} \sqrt{d} & d \equiv 2,3 \pmod 4 \\ \frac{1+\sqrt{d}}{2} & d \equiv 1 \pmod 4 \end{cases} \qquad \{1, \alpha\}\ \mathbb{Z}\text{ basis for } k$$

Consider the homomorphism

$$\sigma = (\sigma_1, \dots, \sigma_d) : k \longrightarrow \mathbb{R}^d \quad , \; \sigma(\alpha) = (\sigma_1(\alpha), \dots, \sigma_d(\alpha))$$

If $\alpha_1, \dots, \alpha_d$ is a basis for $k/\mathbb{Q}$, then $\sigma(\alpha_1), \dots, \sigma(\alpha_d)$ is a basis for $\mathbb{R}^d$ since $\det(\sigma_i(\alpha_j)) \neq 0$. $\therefore$ taking $\alpha_1, \dots, \alpha_d$ is an $\mathbb{Z}$-basis doubt, then see

$\sigma(R_k) = \mathbb{Z}\sigma(\alpha_1) + \dots + \mathbb{Z}\sigma(\alpha_d) < \mathbb{R}^d$ is a lattice : discrete, $\mathbb{R}^d/\sigma(R_k)$ is cpct. [— Special tori from k]

Ex $k = \mathbb{Q}(\sqrt{d})$, $\{1, \alpha\}$ $\mathbb{Z}$-basis, then

$$a + b\alpha \longmapsto (a + b\alpha, a - b\alpha)$$

$d = 2$:

$$a + b\sqrt{2} \longmapsto (a + b\sqrt{2}, a - b\sqrt{2})$$

Now suppose B is a symmetric bilinear form defined over k or R_k.

$$B(u,v) = u^T \beta v \quad \text{where } \beta \in M_{n \times n}(k) \text{ symmetric}$$

Define B^{σ_i} to be given by

$$B^{\sigma_i}(u,v) = u^T \sigma_i(\beta) v \qquad (\sigma_i(\beta))_{k\ell} = (\sigma_i(\beta_{k\ell}))$$

Assume B^{σ_1} has signature $(n-1,1)$ and B^{σ_j} has signature $(n,0)$ or $(0,n)$ $\forall\, j > 1$.

Ex $\beta = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & -\sqrt{d} \end{pmatrix}$ defined over R_k for $k = \mathbb{Q}(\sqrt{d})$ — basis$/\mathbb{R}$: $e_1, \dots, e_{n-1}, \frac{1}{\sqrt[4]{d}} e_n$

signature $(n-1,1)$

$$\beta^{\sigma_2} = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \sqrt{d} \end{pmatrix}$$

signature $(n,0)$

Observe $O(B^{\sigma_i}) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T B^{\sigma_i} A = B^{\sigma_i}\} \cong \begin{cases} O(n-1,1) & i = 1 \\ O(n) & i > 1 \end{cases}$

— change basis.

The homomorphism σ induces a homomorphism

$$\sigma_*: M_{n\times n}(k) \longrightarrow \prod_{i=1}^{d} M_{n\times n}(\sigma_i(k)) \subset \prod_{i=1}^{d} M_{n\times n}(\mathbb{R}) \quad \left(\cong M_{dn\times dn}(\mathbb{R})\right)$$

<u>Note</u> $\sigma_*(M_{n\times n}(R_k))$ is discrete in $\prod_{i=1}^{d} M_{n\times n}(\mathbb{R})$ just as before.

Restricting to $O(B; R_k)$ we have

$$\sigma_*: O(B; R_k) \hookrightarrow \prod_{i=1}^{d} O(B^{\sigma_i}, \sigma_i(R_k)) \subset \prod_{i=1}^{d} O(B^{\sigma_i}, \mathbb{R}) \cong O(n-1,1) \times \prod_{i=2}^{d} O(n)$$

(1) Image $\sigma_*(O(B, R_k))$ is discrete $\Rightarrow$ action on $O(n-1,1)\times \prod O(n)$ is p.d.

(2) $\prod O(n)$ is <u>compact</u>, so $\sigma_*(O(B, R_k))$ also acts P.D. on $O(n-1,1)$ by Proposition B: $O(n-1,1)\times \prod O(n) \longrightarrow O(n-1,1)$

(3) $\therefore$ Image of $O(B, R_k)$ in $O(n-1,1)$ is <u>discrete</u>.

- With a little more work, we can find $\Gamma < O(B, R_k)$ finite index s.t. image lies in $O^+(n-1,1)$ & acts freely.
- With a <u>lot</u> more work one can show $\mathrm{vol}(\mathbb{H}^{n-1}/\Gamma) < \infty$.

(Locally) symmetric spaces. (see Helgason: D.G., Lie gps, Symm. spaces)

Recall from last hw: (M,g) is locally symmetric if $\nabla R = 0$ where R = Riem curv.

Eg (homework) M constant curvature $\Rightarrow$ M locally symmetric (also follows from what we do today)

Before we describe geometric way to think of locally sym. space, we need the following thm of Cartan, that in some sense, implies that the "curvature determines the metric".

Let (M,g), $(\tilde{M},\tilde{g})$ be Riem mfld $m \in M$, $\tilde{m} \in \tilde{M}$. Consider the following map of normal balls:

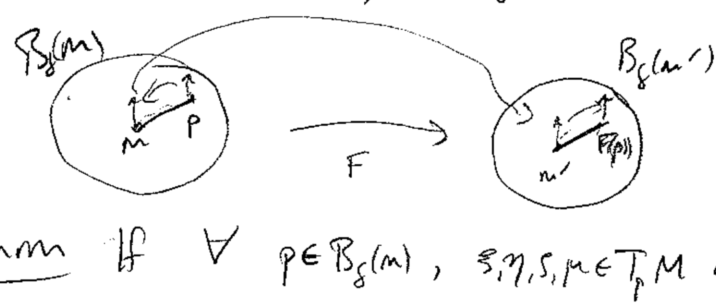
$$F : B_\delta(m) \longrightarrow B_\delta(\tilde{m})$$

$F = \exp_{\tilde{m}} \circ i \circ \exp_m^{-1}$ where $i : T_mM \longrightarrow T_{\tilde{m}}\tilde{M}$ is a linear isometry

$\forall$ $p \in B_\delta(m)$, let $\phi_p : T_pM \longrightarrow T_{F(p)}\tilde{M}$ be given by

$$\phi_p = \tilde{P}_{F(p)} \circ i \circ P_p^{-1} \quad \text{where} \quad P_p : T_mM \longrightarrow T_pM \text{ is parallel transport along}$$

the radial geodesic $\gamma : [0,\ell] \longrightarrow B_\delta(m)$, $\gamma(0) = m$, $\gamma(\ell) = p$. Similarly define $\tilde{P}_{F(p)} : T_{\tilde{m}}\tilde{M} \to T_{F(p)}\tilde{M}$.



Thm If $\forall$ $p \in B_\delta(m)$, $\xi, \eta, \zeta, \mu \in T_pM$ we have

$$g(R(\xi,\eta)\zeta, \mu) = \tilde{g}(\tilde{R}(\phi_p(\xi), \phi_p(\eta))\phi_p(\zeta), \phi_p(\mu))$$

then F is an isometry w/ $dF_p = \phi_p$ $\forall p \in B_\delta(m)$.

Proof:

Let $\gamma : [0, \ell] \to B_\delta(m)$ be geod from m to p, $\xi_1, \dots, \xi_n \in \mathfrak{X}(\gamma)$ parallel o.n. frame over γ, $\xi_n(t) = \gamma'(t)$. Let $J \in \mathfrak{X}(\gamma)$ be a Jacobi field, write

$$J(t) = \sum_{1}^{n} x_i(t)\, \xi_i(t)$$

Jacobi eqn $\Rightarrow$ $\sum_i x_i'' \xi_i + R(\xi_n, J)\xi_n = 0$

dot w/ ξ_j $\Rightarrow$ $x_j'' + g(R(\xi_n, J)\xi_n, \xi_j) \underset{j}{=} 0$

Set $\tilde\xi_1, \dots, \tilde\xi_n \in \mathfrak{X}(\tilde\gamma)$, $\tilde\gamma = F \circ \gamma$, to be

$\tilde\xi_j(t) = \phi_{\gamma(t)}(\xi_j(t))$. By assumpt. $\tilde\xi_j$ are parallel, $\tilde\xi_n(t) = \tilde\gamma'(t)$

Let $\tilde J \in \mathfrak{X}(\tilde\gamma)$ be given by

$$\tilde J(t) = \phi_{\gamma(t)}(J(t)) = \sum x_i(t)\, \tilde\xi_i(t).$$

then

$$\tilde g\left(\frac{D^2 \tilde J}{dt^2} + \tilde R(\tilde\gamma', \tilde J)\tilde\gamma', \tilde\xi_j\right) = x_j'' + \tilde g(\tilde R(\tilde\xi_n, \tilde J)\tilde\xi_n, \tilde\xi_j)$$

$$= x_j'' + g(R(\xi_n, J)\xi_n, \xi_j) = 0.$$

So $\tilde J$ is a Jacobi field.

If $J(0) = 0$ then $\tilde J(0) = 0$ & $J(\ell) = d(\exp_m)_p(\zeta)$, for $\zeta = \ell \frac{DJ}{dt}(0)$

similarly $\tilde J(\ell) = d(\exp_{\tilde m})_{\tilde p}(i(\zeta))$, By the chain rule

$\Rightarrow dF_p(J(\ell)) = \tilde J(\ell)$ $\quad i(\zeta) = i(\ell \frac{DJ}{dt}(0)) = \ell \frac{D\tilde J}{dt}(0).$

and because ϕ_p, $\tilde\phi_{\tilde p}$, i are isometries, $\Rightarrow |dF_p(J(\ell))| = |\tilde J(\ell)| = |J(\ell)|$.

Any vector of T_pM appears as $J(\ell)$, some J, so F is an isometry $\square$.

Gives another alternative proof that any 2 const curvature manifolds of curvature K_0 are locally isometric.

Will use this to prove the following:

(M, g) Riem. $m \in M$. The local symmetry at m is the diffeo.

$\sigma : B_\delta(m) \to B_\delta(m)$ defined on a normal ball given by

$$\sigma(p) = \exp_m(-(\exp_m^{-1}(p)))$$

<u>Thm</u> M is locally symmetric $\iff$ $\forall m \in M$, the local symmetry is an isometry.

<u>Proof</u> suppose first M locally symmetric, so $\nabla R = 0$.

Pick an o.n. frame at m and let $\xi_1, \dots, \xi_n \in \mathfrak{X}(B_\delta(m))$ be the o.n. frame field parallel along radial geodesics, so setting

$$R_{ijkl} = g(R(\xi_i, \xi_j)\xi_k, \xi_l)$$

get, for any radial geodesic γ,

"$\sum \nabla_{\gamma'}$(each factor)" $= 0$

$$\frac{d}{dt}(R_{ijkl}(\gamma(t))) = \gamma'(t) \cdot g(R(\xi_i, \xi_j)\xi_k, \xi_l) = g(\nabla_{\gamma'}(R(\xi_i, \xi_j)\xi_k), \xi_l) + g(R(\xi_i, \xi_j)\xi_k, \nabla_{\gamma'}\xi_l)$$

$$= 0 + 0 = 0$$

R_{ijkl} is constant along radial lines

If we let $\phi_p : T_pM \to T_{\sigma(p)}M$ be defined as above, then clearly

$$\phi_p(\xi_j(p)) = -\xi_j(\sigma(p))$$

So

$$g(R(\xi_i, \xi_j)\xi_k, \xi_l)(p) = R_{ijkl}(p) = R_{ijkl}(\sigma(p)) = g(R(\xi_i, \xi_j)\xi_k, \xi_l)(\sigma(p))$$

$$= g(R(-\xi_i, -\xi_j)-\xi_k, -\xi_l)(\sigma(p)) = g(R(\phi_p(\xi_i)\phi_p(\xi_j))\phi_p(\xi_k), \phi_p(\xi_l))(p)$$

Cartan Thm $\Rightarrow$ σ is an isometry.

For the converse, Fix $m \in M$, $\xi \in T_mM$ extend to parallel orthonormal frame field over geodesic $\gamma : (-\varepsilon,\varepsilon) \to M$, $\gamma(0)=m$, $\gamma'(0)=\xi$: $\xi_1, \dots, \xi_n = \xi \in \mathfrak{X}(\gamma)$

$R_{ijkl}(t) = g(R(\xi_i,\xi_j)\xi_k,\xi_l)(t)$. $\frac{DR}{dt}(0) = \nabla_{\gamma'}R(0) = \nabla_{\xi}R = 0 \iff R_{ijkl}'(0)=0$ $\forall ijkl$.

$$R_{ijkl}'(0) = \lim_{t\to 0} \frac{R_{ijkl}(t) - R_{ijkl}(-t)}{2t} \overset{(*)}{=} 0$$

Note: isometries preserve parallelism $\Rightarrow \xi_i(-t) = d\sigma_{\gamma(t)}(\xi_i(t))$

and preserve curvature

$$R_{ijkl}(-t) = g(R(\xi_i(-t),\xi_j(-t))\xi_k(-t),\xi_l(-t))$$
$$= g(R(d\sigma_{\gamma(t)}(\xi_i(t)), \dots))$$
$$= g(R(\xi_i(t),\xi_j(t))\xi_k(t),\xi_l(t)) = R_{ijkl}(t).$$

So $\nabla R = 0$ □

Corollary product of locally sym. spaces are locally symmetric.

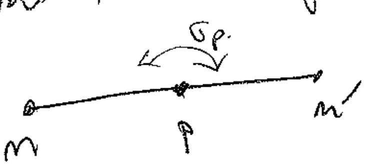
Cor. Const curv. ⇒ loc. sym.

Suppose (M,g) is a complete simply connected locally symmetric space of non pos. curvature. then Hadamard Thm

$\Rightarrow \exp_m : T_mM \to M$ is a diffeo. Previous proof shows

$\forall m$ σ_m, the local sym. w/ $\sigma_m(m)=m$ can be defined on all M and is an isometry — (Globally) symmetric space

It follows that M is homogeneous: • connect any $m, m' \in M$ by a geodesic (Hopf-Rinow).



• p = midpt

• $\sigma_p(m) = m'$.   ⇒ Isom M acts transitively.

[non pos. curvature assumption actually unnecessary]

(M, g) sym. space

$G = \mathrm{Isom}_0(M)$ (compact conn'd)

have the map $F : G \to M$ fibers $\cong K = \mathrm{Stab}_G(m_0)$

Let $\sigma =$ involution fixing m_0, then

$\tau : G \to G$

$\tau(h) = \sigma h \sigma$ $\quad$ σ commutes w/ $K \Rightarrow \tau(K) = K$

Lie subalg of $\mathfrak{g}$ assoc. to K, $\mathfrak{k} < \mathfrak{g}$ is exactly

$$\mathfrak{k} = \{\xi \in \mathfrak{g} \mid (d\tau)_e \xi = \xi\}$$

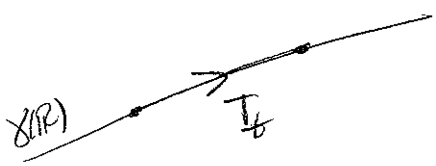
set $\mathfrak{p} = \{\xi \in \mathfrak{g} \mid (d\tau)_e(\xi) = -\xi\}$, then $dF_e|_{\mathfrak{p}} : \mathfrak{p} \xrightarrow{\cong} T_{m_0}M$

(similar to Tues. discussion)

Given $\xi \in \mathfrak{p}$, let $\gamma : \mathbb{R} \to M$ be the geod w/ $\gamma(0) = m$ w/ $\gamma'(0) = dF_e(\xi)$

$\forall\, t \in \mathbb{R}$, let $T_t : M \to M$ be the composition

$$T_t(p) = \sigma_{\gamma(t/2)} \circ \sigma$$



$\forall\, t \in \mathbb{R}$, $T_t(\gamma(\mathbb{R})) = \gamma(\mathbb{R})$ and $T_t(\gamma(0)) = \gamma(t)$ Consider parallel o.n. fields on γ, see

$d(T_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(s+t)}M$ is parallel trans along γ — T_t uniquely determined by this

[using the same translation dist to induce parallel transl.]

$\Rightarrow$ $t \mapsto T_t$ hom of $\mathbb{R} \to G$

$\Rightarrow$ $T_t(p) = \exp(t\xi) \cdot p$

$\exp : \mathfrak{g} \to G$!

So, $t \mapsto \exp(t\xi)p$ is geod through p

- Can also compute curvature for g in terms of $[\,,\,]$ on $\mathfrak{g}$
- Lots more structure!