We want to try to characterize length minimizing paths in a similar way.

Lecture 7

Suppose (M,g) is isometrically embedded in $\mathbb{R}^n$ —

$M \subset \mathbb{R}^n$, $g = g_{std}|_M$.

$\gamma : [a,b] \longrightarrow M \Rightarrow \gamma' : [a,b] \longrightarrow TM$ using $T_m M \subset \mathbb{R}^n \; \forall m$

$\Rightarrow \gamma' : [a,b] \longrightarrow \mathbb{R}^n$ $\left(TM \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ second factor} \right)$
$v_m \longmapsto (m, v_m) \longmapsto v_m$.

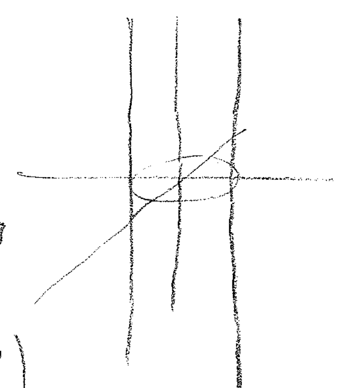
$\gamma'' : [a,b] \longrightarrow \mathbb{R}^n$, no longer necessarily tangent to M.

$\gamma''(t) \overset{?}{\in} T_{\gamma(t)} M$ probably not — $\gamma'(t)$ may be "turning out of M"

Define $\frac{D\gamma'}{dt}(t) = \pi_{\gamma(t)}(\gamma''(t))$ where

$\pi_m : \mathbb{R}^n \longrightarrow T_m M$ is $\perp$-projection

$\frac{D\gamma'(t)}{dt}$ describes how $\gamma'(t)$ is turning, as viewed from M.



EX. $f : M = (-\pi,\pi) \times \mathbb{R} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$f(\theta, z) = (\cos\theta, \sin\theta, z)$.

is isometric embedding for g_{std} on $\mathbb{R}^2$ to g_{std} on $\mathbb{R}^3$

$g_{std}\left(df_{(\theta,z)}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, df_{(\theta,z)}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$ $\qquad df_{(\theta,z)} = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix}$

$= \left\langle \begin{pmatrix} -\sin\theta u_1 \\ \cos\theta u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -\sin\theta v_1 \\ \cos\theta v_1 \\ v_2 \end{pmatrix} \right\rangle = u_1 v_1 + u_2 v_2 = g_{std}\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$

Given $\gamma(t) = \begin{pmatrix} tu_1 \\ tu_2 \end{pmatrix} \Rightarrow (f\circ\gamma)(t) = (\cos(tu_1), \sin(tu_1), tu_2)$, $(f\circ\gamma)'(t) = (-u_1 \sin tu_1, u_1 \cos tu_1, u_2)$

locally length min.

$(f\circ\gamma)''(t) = (-u_1 \cos tu_1, -u_1 \sin tu_1, 0) = -u_1(\cos tu_1, \sin tu_1, 0)$

$\frac{D((f\circ\gamma)')}{dt}(t) = 0$ since $-u_1(\cos tu_1, \sin tu_1, 0) \perp T_{f(\gamma(t))} f(M)$

$$\mathbin{/\!/} \quad \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-\sin(tu_1)\\ \cos(tu_1)\\ 0\end{pmatrix} \right\rangle.$$

More generally, if $\gamma: [a,b] \to M$ is a path, a <u>vector field over γ</u> is a map $\xi: [a,b] \to TM$ s.t. $\xi(t) \in T_{\gamma(t)} M$. If (M,g) is isometrically embedded in $\mathbb{R}^n$ can view $\xi: [a,b] \to \mathbb{R}^n$, define the <u>covariant derivative of ξ</u> to be $\frac{D\xi}{dt}(t) = \pi_{\gamma(t)}(\xi'(t))$

measures how much ξ is turning along γ as view from M in $\mathbb{R}^n$.

More generally still, if $\xi, \eta \in TM$, can measure how ξ is turning in the direction η as viewed from M in $\mathbb{R}^n$:

$\nabla_\eta \xi(m) = \pi_m(d\xi_m(\eta_m))$ where we view $\xi: M \to \mathbb{R}^n$ via $T_m M \hookrightarrow \mathbb{R}^n$.

<u>Rk</u>: Not $L_\eta \xi(m)$ since $\nabla_\eta \xi(m)$ depends only on η_m: we are identify tangent spaces w/ subspaces in $\mathbb{R}^n$ rather than with each other via the flow for η.

Satisfies $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $\forall\, \xi, \eta, \zeta \in \mathfrak{X}(M)$, $f \in C^\infty(M)$

1. $\nabla_{f\xi+\eta} \zeta = f \nabla_\xi \zeta + \nabla_\eta \zeta$
2. $\nabla_\xi (f\zeta + \eta) = \xi(f)\zeta + f \nabla_\xi \zeta + \nabla_\xi \eta$
3. $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$
4. $\xi(g(\zeta, \eta)) = g(\xi(\zeta), \eta) + g(\zeta, \xi(\eta))$ (product rule)

[exercises]

Defn An (affine) connection on M is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M) \qquad \text{denoted } (\xi,\eta) \mapsto \nabla_\xi \eta$$

such that $\forall\, \xi,\eta \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ we have

[some times call ∇ a covariant derivative]

① $\nabla_{f\xi}\eta = f\nabla_\xi \eta$

② $\nabla_\xi f\eta = \xi(f)\eta + f\nabla_\xi \eta$

EX on $\mathbb{R}^n$, $\nabla_\xi \eta(m) = d\eta_m(\xi_m)$ defines a connection.

①&② imply ∇ is local: $\nabla_\xi \eta(m)$ depends only on $\xi(m)$ and $\eta|_U$ for any nbhd U of m. (see Do Carmo p 50, rk 2.3, c.f. localness of $\xi(f)$.)

Given coords $(x_1,\dots,x_n) : U \to \mathbb{R}^n \Rightarrow$ local frame field $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n} = \partial_1,\dots,\partial_n$

$\Rightarrow n^3$ functions $\{\Gamma_{ij}^k\}_{i,j,k=1}^n$ defined by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k .$$

called the Christoffel symbols for ∇

these determine ∇ on U: $\xi = \sum_i \xi^i \partial_i$, $\eta = \sum_j \eta^j \partial_j$

$$\nabla_\xi \eta = \nabla_{\sum \xi^i \partial_i} \sum \eta^j \partial_j = \sum_{i,j} \left(\xi^i \eta^j \nabla_{\partial_i}\partial_j + \xi^i \partial_i(\eta^j)\partial_j \right)$$

$$= \sum_{i,j,k} \xi^i \eta^j \Gamma_{ij}^k \partial_k + \sum_j \xi(\eta_j)\partial_j = \sum_k \left(\left(\sum_{i,j} \xi^i \eta^j \Gamma_{ij}^k \right) + \xi(\eta^k) \right) \partial_k .$$

[← this shows that $\nabla_\xi \eta(m)$ depends only on $\eta|_U$ and ξ_m.]

Given a path $\gamma : [a,b] \to M$ and a v.f. $\xi \in \mathfrak{X}(\gamma)$ can use ∇ to define a covariant derivative, $\frac{D\xi}{dt}$ as follows.

If $\exists$ a v.f. $\hat{\xi} \in \mathfrak{X}(M)$ s.t. $\xi = \hat{\xi} \circ \gamma$, then we want

$$(*) \quad \frac{D\xi}{dt}(t) = \nabla_{\gamma'(t)} \hat{\xi}(\gamma(t))$$

In general, suppose w.l.o.g. $\gamma([a,b]) \subset U$, $x_1,\dots,x_n : U \to \mathbb{R}^n$ coords

$\gamma = (\gamma_1,\dots,\gamma_n)$, $\gamma_i = x_i \circ \gamma$ and write $\xi(t) = \sum_i \xi^i(t)\partial_i|_{\gamma(t)}$

Motivated by $(*)$ define

$$\frac{D\xi}{dt}(t) = \sum_{i,j,k} \frac{d\gamma^i}{dt}(t)\, \xi^j(t)\, \Gamma_{ij}^k(\gamma(t))\partial_k\Big|_{\gamma(t)} + \sum_i \frac{d\xi^i}{dt}(t)\, \partial_i\Big|_{\gamma(t)}.$$

this is well-defined and satisfies:

① $\frac{D}{dt}(\xi+\eta) = \frac{D\xi}{dt} + \frac{D\eta}{dt}$

② $\frac{D}{dt}(f\xi) = \frac{df}{dt}\xi + f\frac{D\xi}{dt}$

unique defn with these properties also satisfying $(*)$.

(see Do Carmo Prop 2.2 p 50-52)

Defn Say $\xi \in \mathfrak{X}(\gamma)$ is <u>parallel along γ</u> if

$$\frac{D\xi}{dt} = 0.$$

<u>Proposition</u> Given $\gamma : [a,b] \longrightarrow M$, $\xi_0 \in T_{\gamma(a)}M$, $\exists!$ parallel v.f.

$\xi \in \mathfrak{X}(\gamma)$ w/ $\xi_0 = \xi(a)$.

<u>proof</u>: W.l.o.g. suppose $\gamma([a,b]) \subset U$, $x_1, \dots, x_n : U \to \mathbb{R}^n$ coords, the required v.f.

$\xi(t) = \sum_i \xi^i(t)\partial_i$ is then precisely a soln of the system of n ODE's.

$$0 = \sum_{i,j}\left[(\Gamma_{ij}^k \circ \gamma)\dot{\gamma}^i\right]\xi^j + \dot{\xi}^k \quad ; \; k = 1, \dots, n \quad \text{w/ init. value } \xi^j(a) = \xi_0^j.$$

This is a linear system so has a unique soln defined for all $t \in [a,b]$ $\square$ (Do Carmo p. 52)

$\forall\ \xi_0 \in T_{\gamma(a)}M$, $t \in [a,b]$ set

$$P_{\gamma,a,t}(\xi_0) = \xi(t)$$

where $\xi \in \mathfrak{X}(\gamma)$ is the ! parallel v.f of the Proposition.

If $c \in \mathbb{R}$, $\xi_0, \eta_0 \in T_{\gamma(a)}M$ and $\xi, \eta \in \mathfrak{X}(\gamma)$ are parallel v.f., then

$$\frac{D}{dt}(c\xi + \eta) = c\frac{D\xi}{dt} + \frac{D\eta}{dt} = 0 \Rightarrow P_{\gamma,a,t}(c\xi_0 + \eta_0) = cP_{\gamma,a,t}(\xi_0) + P_{\gamma,a,t}(\eta_0)$$

So, $P_{\gamma,a,t} : T_{\gamma(a)}M \longrightarrow T_{\gamma(t)}M$ is linear.

Can similarly define $P_{\gamma,t_0,t} : T_{\gamma(t_0)}M \to T_{\gamma(t)}M \quad \forall\ t_0, t \in [a,b]$

and verify $P_{\gamma,t_0,t_1} \circ P_{\gamma,t_1,t} = P_{\gamma,t_0,t}$, so each $P_{\gamma,t_1,t}$ is an isomorphism.

$P_{\gamma,t_0,t}$ is called the <u>parallel transport along γ</u>.

Homework exercise:

$\xi, \eta \in \mathfrak{X}(M)$, $\gamma : [a,b] \to M$ an integral curve of ξ, $\gamma(0) = p$ then

$$\nabla_\xi \eta(p) = \frac{d}{dt}\Big|_{t=0} \left(P_{\gamma,0,t}^{-1} (\eta(\gamma(t))) \right)$$

looks similar to Lie derivative except instead of using flow to pull η back, we use parallel transport. More later...

Lecture 8

(M, g) a Riem. mfd, ∇ a connection.

<u>Defn</u> ∇ is <u>compatible w/ g</u> if $\forall\ \xi, \eta, \zeta \in \mathfrak{X}(M)$

$$\xi(g(\eta,\zeta)) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta). \quad \text{[product rule for inner product]}$$

Equivalently, $\forall\ \gamma : [a,b] \to M$, $\xi, \eta \in \mathfrak{X}(\gamma)$ parallel vector fields over γ we have $g(\xi, \eta)$ is constant (as a function of t). (See DC p.53, 54)

Alternatively, using parallel transport, we can define covariant derivatives of tensor fields and diff'l forms. Then

g is compatible with $\nabla \iff \nabla_\xi g = 0. \ \forall\ \xi \in \mathfrak{X}(M)$

Covariant derivatives of tensor field and diff'l forms:

Ex If α is a $(0,p)$-tensor or diff'l p-form, then we can define, for $\xi \in \mathfrak{X}(M)$, the covariant derivative

$$\nabla_\xi \alpha(m) = \frac{d}{dt}\Big|_{t=0} P^*_{\gamma,0,t}(\alpha_{\gamma(t)})$$

where $\gamma : (-\varepsilon,\varepsilon) \to M$ is an integral curve for ξ w/ $\gamma(0)=m$.

depends only on ξ_m and α in a nbhd of m, satisfies

$$\nabla_\xi(\alpha + f\beta) = \nabla_\xi \alpha + \xi(f)\beta + f\nabla_\xi \beta, \ldots$$

Easist to show all properties by proving that this satisfies (and is in fact determined by) the equation

$$\xi\big(\alpha(\eta_1,\ldots,\eta_p)\big) = (\nabla_\xi \alpha)(\eta_1,\ldots,\eta_p) + \sum_i \alpha(\eta_1,\ldots,\nabla_\xi \eta_i,\ldots,\eta_p)$$

This is a kind of product rule — we view $\alpha(\eta_1,\ldots,\eta_p)$ as a product of $\alpha,\eta_1,\ldots,\eta_p$. Other product rules that we can verify:

$$\nabla_\xi(\alpha\otimes\beta) = (\nabla_\xi\alpha)\otimes\beta + \alpha\otimes\nabla_\xi\beta, \quad \nabla_\xi(\alpha\wedge\beta) = (\nabla_\xi\alpha)\wedge\beta + \alpha\wedge(\nabla_\xi\beta).$$

Ex If g is a Riem. metric and ∇ a connection, we have

$$\xi(g(\eta,\zeta)) = (\nabla_\xi g)(\eta,\zeta) + g(\nabla_\xi\eta,\zeta) + g(\eta,\nabla_\xi\zeta) \quad \forall \xi,\eta,\zeta \in \mathfrak{X}(M)$$

So, g is compatible w/ $\nabla \iff \nabla_\xi g = 0 \ \forall \xi \in \mathfrak{X}(M)$, as stated above.

<u>Defn</u> A connection ∇ on M is <u>symmetric</u> (or <u>torsion free</u>)
if $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta] \quad \forall\, \xi, \eta \in \mathfrak{X}(M)$.

Equiv, in coords, $\Gamma^k_{ij} = \Gamma^k_{ji} \quad \forall\, i, j$.

Alternatively, the <u>torsion</u> of ∇ is defined to be the <u>tensor</u>

$$T_\nabla(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta].$$ [check - it really is a tensor]

Then ∇ is torsion-free $\Leftrightarrow T_\nabla = 0$.

<u>Theorem</u> (Levi-Civita) Given (M, g) $\exists!$ connection ∇ on M that is compatible w/ g and symmetric

[So, for $M \subset \mathbb{R}^n$ we obtain the motivating connection uniquely boring].

<u>proof</u> Assume existence, we verify uniqueness.

$$\xi(g(\eta, \zeta)) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta)$$

$$+ \quad \eta(g(\zeta, \xi)) = g(\nabla_\eta \zeta, \xi) + g(\zeta, \nabla_\eta \xi)$$

$$- \quad \zeta(g(\xi, \eta)) = g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta)$$

$$\xi(g(\eta, \zeta)) + \eta(g(\zeta, \xi)) - \zeta(g(\xi, \eta)) = g(\eta, [\xi, \zeta]) + g(\xi, [\eta, \zeta]) + g(\zeta, [\eta, \xi]) + 2g(\zeta, \nabla_\xi \eta)$$

one term w/ ∇ - others determine $\nabla \Rightarrow$ uniqueness.

Alt., use this to define ∇ $\square$

Worth writing this in coords. $x_1, \dots, x_n : U \to \mathbb{R}^n$

$$g(\partial_i, \partial_j) = g_{ij}$$

Then, since $[\partial_i, \partial_j] = 0$, we get.

$$2g(\nabla_{\partial_i}\partial_j, \partial_k) = \partial_i(g(\partial_j, \partial_k)) + \partial_j(g(\partial_k, \partial_i)) - \partial_k(g(\partial_j, \partial_i))$$

or

$$g\left(\sum_\ell \Gamma_{ij}^\ell \partial_\ell, \partial_k\right) = \tfrac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right)$$

or

$$\sum_\ell \Gamma_{ij}^\ell g_{\ell k} = \tfrac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right)$$

$(g_{\ell k})$ is invertible, w/ inverse (g^{kr}) — so $\sum_k g_{\ell k} g^{kr} = \delta_\ell^r$ and

$$\sum_{k,\ell} \Gamma_{ij}^\ell g_{\ell k} g^{kr} = \sum_k g^{kr} \tfrac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right)$$

or

$$\Gamma_{ij}^r = \quad ''$$

EX in std coords on $\mathbb{R}^n$, w/ $g = g_{std}$, $g_{ij} = \delta_{ij}$

$\Rightarrow \Gamma_{ij}^k \equiv 0$.

EX Homework – compute L-C connection for $\mathbb{H}^n$: behaves just like $M \subset \mathbb{R}^n$.

EX $U \subset \mathbb{R}^n$, define a new metric on U by scaling g_{std} by a positive function. Convenient to write

$$g = \frac{1}{F^2} g_{std} \qquad \text{for } F \in C^\infty(U) \text{ a positive function}$$

Then $g_{ij} = \frac{1}{F^2}\delta_{ij}$ and inverse is $g^{ij} = F^2\delta^{ij}$.

Setting $f = \log(F)$ we have $\frac{\partial_k F}{F} = \partial_k f = f_k$

$$\partial_k g_{ij} = \partial_k\left(\tfrac{1}{F^2}\right)\delta_{ij} = \frac{-2}{F^3}\partial_k F\,\delta_{ij} = \frac{-2}{F^2} f_k\,\delta_{ij}.$$

so
$$\Gamma_{ij}^k = \frac{1}{2}\sum_{\ell}\left(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}\right) g^{\ell k} = \frac{F^2}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right)$$
$$= -\left(f_i\delta_{jk} + f_j\delta_{ki} - f_k\delta_{ij}\right)$$

∴ $\Gamma_{ij}^k = 0$ if i,j,k are distinct, while in the remaining cases:

$$\underset{i\neq j}{\Gamma_{ij}^i = \Gamma_{ji}^i = -f_j}, \quad \underset{i\neq k}{\Gamma_{ii}^k = f_k}, \quad \Gamma_{ii}^i = -f_i$$

<u>Ex</u> Another model of $\mathbb{H}^n$: $F = x_n$ on $U^n = \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_n > 0\}$.

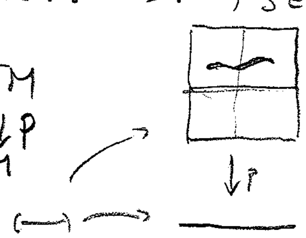
$$f_j = \partial_j(\log(x_n)) = \frac{\delta_{jn}}{x_n}.$$

<u>Ex</u> Use stereographic projection $f: S^n - \{(1,0,\dots,0)\} \longrightarrow \mathbb{R}^n$, $f_*(g) = \frac{1}{F^2} g_{std}$.
exercise – Find F.

Another perspective on connections: M, ∇. [see Ch 3, HW 2]

Given a curve $\gamma: [a,b] \longrightarrow M$, $\xi \in \mathfrak{X}(\gamma)$ is a <u>lift</u> of γ to TM:

$$\begin{tikzcd} & TM \arrow[d, "p"] \\ {[a,b]} \arrow[ur, "\xi"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$



parallel transport gives preferred lifts with any starting point $v \in T_{\gamma(a)}M$

$t \longmapsto P_{\gamma,a,t}(v)$. Write $\gamma_v(t) = P_{\gamma,a,t}(v)$.

Can use this to split the tangent space of TM —

given $v_m \in T_mM \subset TM$, consider $T_{v_m}(TM)$.

There is a canonically defined $\frac{1}{2}$-dim'l subspace, called

the <u>vertical space</u>: $V_{v_m} = \ker(dp_{v_m}) = T_{v_m}(T_mM) \cong T_mM$

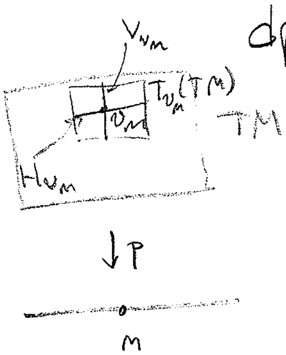
$\uparrow$ canonical since T_mM is a vector space

A <u>horizontal space</u> $H_{v_m} \subset T_{v_m}(TM)$ is a

complement to V_{v_m}: $T_{v_m}(TM) = V_{v_m} \oplus H_{v_m}$.

Note: If H_{v_m} is a horizontal space, then

$dp_{v_m}\big|_{H_{v_m}} : H_{v_m} \to T_mM$ is an $\cong$ by dimension consideration.



A connection determines such a subspace by

defining the inverse isomorphism

$$T_mM \xrightarrow{\cong} H_{v_m} \subset T_{v_m}(TM)$$

by $\quad \gamma'(0) \longmapsto \gamma_{v_m}'(0) \quad$ where γ_{v_m} is the parallel v.f. over γ starting at v_m

This choice of $H_{v_m} \subset T_{v_m}(TM)$ for all $v_m \in TM$ is precisely a distribution on TM which we call the horizontal distribution of ∇, and denote it H^∇, w/ $H^\nabla_{v_m} \subset T_{v_m}(TM)$.

Not obviously a distribution, but we can alternatively define it as follows.

We construct a map $\Pi^\nabla : TTM \to TM$ s.t.

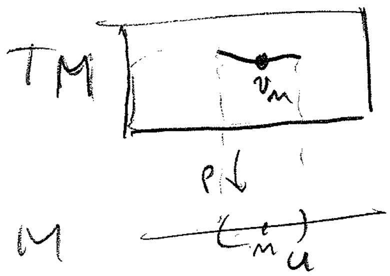
① $\Pi^\nabla\big|_{T_{v_m}(TM)} = \Pi^\nabla_{v_m} : T_{v_m}(TM) \to T_mM$ is linear

② $\Pi^{\nabla}_{v_m}\big|_{V_{v_m}} : V_{v_m} \longrightarrow T_m M$ is the canonical isomorphism.

Then $H^{\nabla}_{v_m} = \ker(\Pi^{\nabla}_{v_m})$.

To define this, need some way to describe $T_{v_m}(TM)$:

Let $\xi \in \mathcal{X}(U)$ be a v.f. on a nbhd U of m w/ $\xi_m = v_m$



this is an embedding, since $p \circ \xi : U \to U$ is the identity, $(d\xi)_m : T_m M \to T_{v_m}(TM)$ is an isomorphism onto a horizontal subspace.

define $\Pi^{\nabla}_{v_m}((d\xi)_m(u_m)) := \nabla_{u_m} \xi(m)$

use loc coords to check well defined.

By construction, covariant derivatives are ordinary derivatives composed w/ Π^{∇}:

$$\nabla_\xi \eta(m) = \Pi^{\nabla}_{\eta_m}((d\eta)_m(\xi_m)) \quad \& \quad \frac{D\xi}{dt}(t) = \Pi^{\nabla}_{\xi(t)}\left(\frac{d\xi}{dt}(t)\right)$$

parallel vector fields $\xi \in \mathcal{X}(\gamma)$ over $\gamma : [a,b] \to M$ are precisely those lifts of γ w/ horizontal tangent vectors:

$\xi \in \mathcal{X}(\gamma)$ is parallel $\iff \xi'(t) \in H^{\nabla}_{\xi(t)}\ \forall t$.

Exercise: If $x_1, \dots, x_n : U \to \mathbb{R}^n$ are coords, get coords.

$x_1, \dots, x_n, y_1, \dots, y_n : TU \to \mathbb{R}^n \times \mathbb{R}^n$ $\quad \langle \frac{\partial}{\partial x_i}\big|_{(x,y)} \rangle$ is a horiz. distrib

$\langle \frac{\partial}{\partial y_i}\big|_{(x,y)} \rangle = V_{v_m}$. Check H^{∇} is defined by

$$\Pi^{\nabla}_{(x,y)}\left(\frac{\partial}{\partial y_i}\Big|_{(x,y)}\right) = \frac{\partial}{\partial x_i}\Big|_x \text{ and } \Pi^{\nabla}_{(x,y)}\left(\frac{\partial}{\partial x_i}\Big|_{(x,y)}\right) = \sum_{j,k} y^j \Gamma^k_{ij}(x) \frac{\partial}{\partial x_k}\Big|_x$$

Lecture 9

We now return to the motivating situation.

Ch 3 M a mfd w/ connection ∇, $I \subset \mathbb{R}$ an interval

Defn $\gamma: I \to M$ is <u>geodesic</u> at t if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)(t) = 0$.

If γ is geod. at every t, then say γ is a <u>geodesic</u>.

[Rk: In metric geometry a geodesic is an isometric embedding, we'll see that this is related, but not quite the same]

<u>Existence of geodesics</u>:

We construct a vector field $G \in \mathfrak{X}(TM)$ for which the flow lines project to (all) geodesics in M. Let $v_m \in TM$ and

Set G_{v_m} = horizontal lift of v_m

i.e. ① $G_{v_m} \in H^{\nabla}_{v_m}$ w/ ② $dp_{v_m}(G_{v_m}) = v_m$.

Let Φ_t be the local flow and $\sigma(t, v_m) = \Phi_t(v_m)$.

Consider $\gamma(t) = p(\sigma(t, v_m))$

then $\gamma'(t) = dp_{\sigma(t,v_m)}(\sigma'(t,v_m)) = dp_{\sigma(t,v_m)}(G_{\sigma(t,v_m)}) = \sigma(t, v_m)$ by ②

so, $\gamma' \in \mathfrak{X}(\gamma)$ is exactly $\sigma(t, v_m)$ (in particular, $\gamma'(0) = v_m$)

Then $\frac{D\gamma'}{dt}(t) = \Pi^{\nabla}_{\sigma(t,v_m)}(\sigma'(t,v_m)) = \Pi^{\nabla}_{\sigma(t,v_m)}(G_{\sigma(t,v_m)}) = 0$ by ①.

So $\gamma(t)$ is a geodesic. Conversely, check that γ a geodesic, then $\gamma'(t)$ is a flow line of G. So we have shown:

Proposition $\forall\, m \in M$, $v_m \in T_mM$ $\exists!$ max. geod. $\gamma(t, v_m) = p(\sigma(t, v_m))$ with $\gamma(0, v_m) = m$, $\gamma'(0, v_m) = v_m$.

Note: $\forall\, s \in \mathbb{R}_+$ we have homogeneity:

$$\gamma(t, s v_m) = \gamma(st, v_m)$$

since $\frac{d}{dt}(\gamma(st, v_m)) = s\gamma'(st, v_m)$ and $\frac{D}{dt}\left(\frac{d}{dt}(\gamma(st, v_m))\right) = \frac{D}{dt}(s\gamma'(st, v_m))$

$$= s\frac{D}{dt}(\gamma'(st, v_m))$$

and $\frac{d}{dt}(\gamma(st, v_m))\Big|_{t=0} = s\gamma'(st, v_m) = s v_m$.

$$= s^2 \frac{D\gamma'}{dt}(st, v_m) = 0$$

So, since $\Phi_t(0_m)$ is defined $\forall\, t, m$ we can find a nbhd

$\mathcal{U}$ of 0_m in TM s.t. $\Phi_t(v_q)$ is defined $\forall\, t \in (-2, 2)$, $q \in p(\mathcal{U})$.

$\Phi : (-2, 2) \times \mathcal{U} \longrightarrow TM$.

If ∇ is the Levi-Civita connection for g we can assume

$\mathcal{U} = \{ v_q \mid q \in U \underset{\text{open}}{\subseteq} M,\ |v_q| < \varepsilon \}$ (here $|v_q| = \sqrt{g_q(v_q, v_q)}$)

Def'n With $\mathcal{U}$ as above, define the exponential map on $\mathcal{U}$

$$\exp : \mathcal{U} \to M$$

by $\exp(v_q) = p(\Phi_1(v_q)) = \gamma(1, v_q) = \gamma\left(|v_q|, \frac{v_q}{|v_q|}\right)$ by homogeneity.

at each $q \in p(\mathcal{U}) = U$ we can write

$$\exp_q = \exp\big|_{T_qM} : B_\varepsilon(q) = \{ v_q \in T_q M \mid |v_q| < \varepsilon \} \longrightarrow M$$

by $\exp_q(v_q) = \exp(v_q)$. Composition of smooth, so smooth. In fact:

Theorem $\exp_q$ is a diffeomorphism from a nbhd of 0 to an open set in M. Indeed, $d(\exp_q)_0 = id_{T_qM}$.

proof: 1^{st} follows from 2^{nd} and I.F.T. For the second, compute

$$d(\exp_q)_0(v_q) = \frac{d}{dt}\Big|_{t=0}(\exp_q(t v_q)) = \frac{d}{dt}\Big|_{t=0}\left(\gamma\left(t|v_q|, \frac{v_q}{|v_q|}\right)\right) = |v_q|\,\gamma'\left(t|v_q|, \frac{v_q}{|v_q|}\right) = v_q.$$

$\square$

Minimizing Properties of geodesics:

We'll see that geodesics are locally length minimizing in the sense that if $\gamma : I \longrightarrow M$ is a unit speed geodesic then $\forall\ t_0 \in I$, $\exists\ \varepsilon > 0$ st.

$$d(\gamma(t), \gamma(s)) = |t-s| \quad \forall\ t, s \in (t_0 - \varepsilon, t_0 + \varepsilon)$$

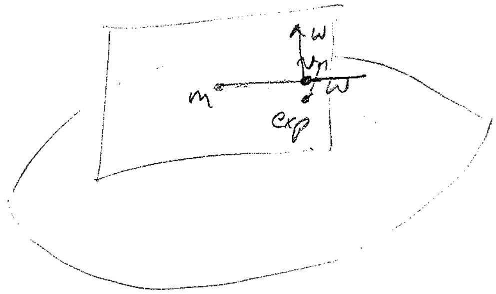
Further more, any path $\gamma : I \to M$ w/ this property is a geodesic.

[Key tool needed to prove this is the map $\exp_m$. To make use of this we need the following technical fact]

Lemma (Gauss) Let $m \in M$, $v \in T_m M$ st. $\exp_m v$ is defined.

Given $w \in T_m M \cong T_v(T_m M)$, we have

$$\langle (d\exp_m)_v(v), (d\exp_m)_v(w) \rangle_{\exp_m(v)} = \langle v, w \rangle_m.$$



Easy part:

Note $\exp_m(t \frac{v}{|v|}) = \gamma(t, \frac{v}{|v|})$ is a unit speed geodesic (call these radial geodesics from m)

so $(d\exp_m)_v(\frac{v}{|v|}) = \frac{d}{dt}\Big|_{t=|v|} \left(\exp_m (t \frac{v}{|v|}) \right) = \gamma'(|v|, \frac{v}{|v|})$

$\Rightarrow |(d\exp_m)_v(v)| = |v| \, |\gamma'(|v|, \frac{v}{|v|})| = |v| \cdot 1 = |v|$.

So, on each ray, $\exp_m$ is an isometric embedding.

$\therefore$ Gauss Lemma is equivalent to saying:

$$w \perp v, \text{ then } (d\exp_m)_v(v) \perp (d\exp_m)_v(w)$$

Assume this for a moment and see how to use it.