Theorem. *If* $\Gamma < Isom^+(\mathbb{H}^2)$ *is a cocompact Fuchsian group, then*

$$\mu\left(\mathbb{H}^2/\Gamma) \geq \frac{\pi}{21}\right)$$

Proof. Writing the signature $\mathrm{sign}(\Gamma) = (g; c_1, ..., c_N)$, then the Gauss-Bonnet Theorem states:

$$\mu(\mathbb{H}^2/\Gamma) = 2\pi\left(2g - 2 + \sum_{i=1}^{N}\left(1 - \frac{1}{c_i}\right)\right)$$

Note: $1 - \frac{1}{c_i} \geq \frac{1}{2}$ for each i with equality if and only if $c_i = 2$.

Further, if $\mathrm{sign}(\Gamma) = (g'; c_1', ..., c_N')$ and $g \leq g'$, $c_i \leq c_i'$ for each i, then

$$\mu(\mathbb{H}^2/\Gamma) \leq \mu(\mathbb{H}^2/\Gamma')$$

We now consider all possibilities for the signature. For each signature, we either bound the area from below by $\pi/21$ or else verify that there is no hyperbolic orbifold with that signature.

- $\mathbf{g \geq 2}$:

  $$\mu(\mathbb{H}^2/\Gamma) \geq 2\pi(2g-2) \geq 4\pi$$

- $\mathbf{g = 1}$:

  $$\mu(\mathbb{H}^2/\Gamma) = 2\pi\left(\sum_{i=1}^{N}\left(1 - \frac{1}{c_i}\right)\right)$$

  This is positive if and only if $N > 0$. In this case

  $$\mu(\mathbb{H}^2/\Gamma) \geq 2\pi\frac{1}{2} = \pi$$

- $\mathbf{g = 0}$:
  - $\mathbf{N \geq 5}$: $$\mu(\mathbb{H}^2/\Gamma) = 2\pi\left(-2 + \sum_{i=1}^{N}\left(1 - \frac{1}{c_i}\right)\right) \geq 2\pi\left(-2 + \frac{5}{2}\right) = \pi$$
  - $\mathbf{N = 4}$: $$\mu(\mathbb{H}^2/\Gamma) = 2\pi\left(-2 + \sum_{i=1}^{4}\left(1 - \frac{1}{c_i}\right)\right) = 2\pi\left(2 - \sum_{i=1}^{4}\frac{1}{c_i}\right)$$

    This area is positive if and only if $(c_1, c_2, c_3, c_4) \neq (2, 2, 2, 2)$. Furthermore it is minimized when $(c_1, c_2, c_3, c_4) = (2, 2, 2, 3)$:

    $$\mu(\mathbb{H}^2/\Gamma) = 2\pi\left(2 - \left(\frac{3}{2} + \frac{1}{3}\right)\right) = \frac{\pi}{3}$$
  - $\mathbf{N = 3}$: $$\mu(\mathbb{H}^2/\Gamma) = 2\pi\left(1 - \sum_{i=1}^{3}\frac{1}{c_i}\right)$$

    For this to be positive, we need

    $$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} < 1$$

    This means (c_1, c_2, c_3) can **not** be of one of the following types:

    $$(2, 2, n)\,,\ n \geq 2\,; \quad (2, 3, n)\,,\ n = 3, ..., 6; \quad (2, 4, 4)\,; \quad (3, 3, 3)$$

    The minimal triples (c_1, c_2, c_3) remaining, along with the area of their respective orbifolds are given by

    $$(2, 3, 7)\,,\ \frac{\pi}{21}\,; \quad (2, 4, 5)\,,\ \frac{\pi}{10}\,; \quad (3, 3, 4)\,,\ \frac{\pi}{6}$$
  - $\mathbf{N \leq 2}$: There are no hyperbolic orbifolds in this case.

This gives all the cases and so completes the proof. □