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ESSENTIAL SURFACES IN HYPERBOLIC THREE-MANIFOLDS

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ESSENTIAL SURFACES IN HYPERBOLIC THREE-MANIFOLDS

by

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To my wife and my parents.

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ESSENTIAL SURFACES IN HYPERBOLIC THREE-MANIFOLDS

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This dissertation is concerned with existence and behavior problems for essential surfaces in hyperbolic three-manifolds. In Chapter 2, we provide new examples of finite volume hyperbolic manifolds which are not fibered, but are virtually fibered. In Chapter 3, we prove that there are Dehn fillings arbitrarily close to infinity in the hyperbolic Dehn surgery space of the figure eight knot complement in which some closed totally geodesic surface compresses.

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Chapter 1

Introduction

A proper map

$$f : F \to M$$

of a compact, orientable, connected surface ($F \not\cong S^2$) into an irreducible three-manifold M is called *incompressible* if

$$f_* : \pi_1(F) \to \pi_1(M)$$

is injective, (or $F \cong D^2$ and ∂F is essential in ∂M). If f is not properly homotopic into ∂M, then $f : F \to M$ is *essential* in M (briefly, F is essential).

In the development of 3-manifold topology, essential embedded surfaces have played a central role. For example, irreducible 3-manifolds which contain embedded incompressible surfaces are called *Haken*. Haken manifolds also contain embedded essential surfaces, and can be effectively analyzed by an inductive procedure. This was exploited by Thurston who proved that if a Haken manifold contains no essential tori, then it is hyperbolic (with one obvious exception) [41].

More recently the study of immersed essential surfaces has become important. This is due in part to Thurston's conjectural picture of finite volume hyperbolic 3-manifolds. In [41], Thurston asked the following question, which he conjectured to have an affirmative answer.

Question 1.0.1. *Is every finite volume hyperbolic 3-manifold finitely covered by a surface bundle over the circle?*

When M is finitely covered by a surface bundle over the circle, the fiber pushes down to an immersed essential surface in M, and this question thus becomes a question about immersed essential surfaces. Despite the conjectural positive answer to this question, constructing non-trivial examples of 3-manifolds providing evidence for this answer seems difficult in general. In Chapter 2, we construct a new class of examples supporting this conjecture.[1] In particular, we construct what seem to be the first knot exteriors in S^3 which are not fibered, but are virtually fibered.

Another problem involving essential surfaces in a hyperbolic 3-manifold is concerned with the behavior of such surfaces after performing Dehn filling. Specifically, if M is hyperbolic with non-empty torus boundary, Dehn filling on one or more components of ∂M constructs new manifolds having M as a submanifold. Deciding when an essential surface, F, in M remains essential after Dehn filling is a question which has received much attention in the past (see e.g. [5], [4], or [2]). In the case that F is embedded and has no accidental parabolics, very precise information is available concerning the Dehn fillings of M in which F can compress (see [5] and [42]). In Chapter 3, we consider

[1]Portions of Chapter 2 reprinted from [20] with permission from Elsevier Science.

the analogous problem where the embedded hypothesis has been removed.[2] We prove that this hypothesis is necessary to retain the same information on incompressibility. In fact, we show that without this hypothesis, the situation is as bad as possible.

1.1 Some preliminary material

We begin by recalling a few definitions and constructions to fix notation.

1.1.1 Dehn filling and orbifolds

Here we collect some of the basic facts and definitions concerning 3-manifolds and orbifolds, and Dehn filling, see [15], [36], [27], and [39] for more details.

Let M denote a compact, orientable 3-manifold or 3-orbifold with a torus boundary component, $\partial_0 M \cong T^2$ (which we assume is disjoint from the singular locus for orbifolds). A *slope* on $\partial_0 M$ is an isotopy class of unoriented essential simple closed curves on $\partial_0 M$. The *distance* between slopes α and β, denoted $\Delta(\alpha, \beta)$, is defined to be the absolute value of their algebraic intersection number (given arbitrary orientations). Slopes on $\partial_0 M$ are in a 2 to 1 correspondence with primitive elements of $\pi_1(\partial_0 M) \cong H_1(\partial_0 M; \mathbb{Z}) \cong \mathbb{Z}^2$, with the ambiguity coming from the lack of orientations. We will often ignore this ambiguity, making no distinction between slopes and primitive elements of $H_1(\partial_0 M; \mathbb{Z})$.

Let μ and λ denote generators for $H_1(\partial_0 M; \mathbb{Z})$. Slopes on $\partial_0 M$ then correspond to co-prime integer pairs in $\mathbb{Z}^2$ by associating the pair (p, q) to

[2]Portions of Chapter 3 reprinted from [19] with permission from Elsevier Science.

the slope $p\mu + q\lambda$ (note that (p, q) and $(-p, -q)$ represent the same slope). One checks that if $\alpha = (p, q)$ and $\beta = (p', q')$ are two slopes, then $\Delta(\alpha, \beta) = |pq' - p'q|$.

Given a slope α on $\partial_0 M$, one can form a new orbifold (or manifold, when M is a manifold) $M(\alpha)$ by *α-Dehn filling* on $\partial_0 M$, as follows. Let $S^1 \times D^2$ be a solid torus. Choosing a homeomorphism $h : \partial(S^1 \times D^2) \to \partial_0 M$, so that $h(* \times \partial D^2)$ represents α, we can glue $S^1 \times D^2$ to M by identifying points x and $h(x)$. The resulting space is an orbifold (or manifold), and up to homeomorphism, depends only on α. If we have chosen a basis μ, λ for $H_1(\partial_0 M; \mathbb{Z})$ and α is given by (p, q), we denote $M(\alpha)$ by $M(p, q)$. Note that there is a natural inclusion $i : M \to M(\alpha)$.

A variation of this construction that we will make use of is *orbifold Dehn filling*, which we now describe. Given an integer $d > 1$ and a slope α on $\partial_0 M$, we first construct $M(\alpha)$. The new orbifold, denoted $M(d\alpha)$, is gotten by giving the core curve, $S^1 \times \{0\}$, in the filling solid torus a transverse angle of $2\pi/d$, making it (part of) the singular locus with local group $\mathbb{Z}/d\mathbb{Z}$. We say that $M(d\alpha)$ is obtained from M by *$d\alpha$-orbifold Dehn filling.* As above, if μ, λ is a basis for $H_1(\partial_0 M; \mathbb{Z})$, and α is given by (p, q), we denote $M(d\alpha)$ by $M(dp, dq)$.

One of the primary examples of compact, orientable 3-manifolds with toroidal boundary which we will make use of is the exterior of a link in S^3. Given a (tame) link $L \subset S^3$ we let $N(L)$ denote an open tubular (or regular) neighborhood of L. The *exterior of L in S^3* is given by

$$X(L) = S^3 \setminus N(L).$$

$\partial X(L)$ is a disjoint union of tori.

Pillow case

An example of a 2-orbifold which we will encounter frequently is the *pillow case*, which is obtained as the quotient of a torus T by a hyperelliptic involution. We may view this as an order 2 rotation about a line l in $\mathbb{R}^3$, restricted to T, where T is embedded in $\mathbb{R}^3$ so that it intersects l in exactly 4 points (see Figure 1.1).

The pillow case has a 2-sphere as its underlying space, with the singular locus consisting of 4 points, each having local group $\mathbb{Z}/2\mathbb{Z}$.

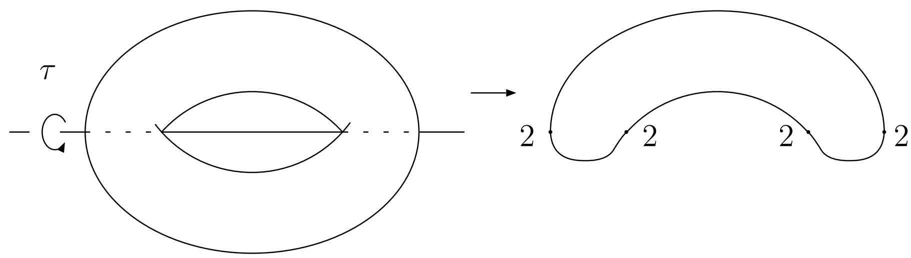

Figure 1.1: Involution giving the pillow case.

Commensurability

Two subgroups $H, K < G$ are said to be *commensurable* in G if there exists $g \in G$ such that $H \cap gKg^{-1}$ is a subgroup of finite index in both H and gKg^{-1}. Two orbifolds (or manifolds) are *commensurable* if they share a finite sheeted orbifold covering space. In both cases, commensurability is easily seen to be an equivalence relation.

The following gives a simple criteria for two orbifolds to be commensurable.

Lemma 1.1.1. *If M and N are finite sheeted orbifold covers of an orbifold P, then M and N are commensurable.*

Proof. By hypothesis, $\pi_1^{orb}(M)$ and $\pi_1^{orb}(N)$ inject into $\pi_1^{orb}(P)$, both with finite index. The intersection of their images is a finite index subgroup of $\pi_1^{orb}(P)$. The corresponding orbifold is a finite sheeted covering of both M and N. □

1.1.2 Hyperbolic geometry

We review a few of the basics of hyperbolic geometry. For more details, see [3],[31], or [39].

Hyperbolic space and its isometries

We will primarily use the upper half space model for hyperbolic 3-space, $\mathbb{H}^3$. That is,

$$\mathbb{H}^3 = \{(z,t) \,|\, z = x + iy \in \mathbb{C}\,,\, t > 0\}$$

with the complete Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

$\mathbb{H}^3$ is compactified by $S^2_\infty = \widehat{\mathbb{C}}$ and all orientation preserving isometries are conformal extensions of conformal maps of $\widehat{\mathbb{C}}$. Thus, $PSL_2(\mathbb{C})$ is the full group of orientation preserving isometries of $\mathbb{H}^3$, acting by extension of linear fractional transformations on $\widehat{\mathbb{C}}$.

Similarly, the upper half plane model of the hyperbolic plane is

$$\mathbb{H}^2 = \{(x,t) \,|\, x \in \mathbb{R}\,,\, t > 0\}$$

with metric

$$ds^2 = \frac{dx^2 + dt^2}{t^2}.$$

$\mathbb{H}^2$ is naturally compactified by $S^1_\infty = \widehat{\mathbb{R}}$. $PSL_2(\mathbb{R})$ is the full group of orientation preserving isometries of $\mathbb{H}^2$ acting by extension of linear fractional transformations on $\widehat{\mathbb{R}}$. Note that we can view $\mathbb{H}^2$ as a submanifold of $\mathbb{H}^3$ embedded totally geodesically. When this is done, the action of $PSL_2(\mathbb{R})$ on $\mathbb{H}^2$ is the restriction to $\mathbb{H}^2$ of the action of $PSL_2(\mathbb{R}) \subset PSL_2(\mathbb{C})$ on $\mathbb{H}^3$. When we identify $\mathbb{H}^2$ with the upper half plane of $\mathbb{C}$ (in the obvious way), the action of $PSL_2(\mathbb{R})$ is by linear fractional transformations.

Remark 1.1.1. Let $P : SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$ denote the quotient map. Whenever we refer to a matrix for an element $g \in PSL_2(\mathbb{C})$, we mean a matrix $A \in P^{-1}(g)$.

Given $\gamma \in PSL_2(\mathbb{C})$, the trace of γ, $Tr(\gamma)$, is well defined up to sign. We say that γ is *elliptic* if $Tr(\gamma) \in (-2, 2)$, *parabolic* if $Tr(\gamma) = \pm 2$, and *hyperbolic* otherwise. For $\gamma \neq 1$, then $Tr(\gamma)$ is a complete invariant of the conjugacy class of γ. See [39] for a geometric description of the action on $\mathbb{H}^n$, $n = 2, 3$.

Kleinian groups

A discrete subgroup $\Gamma \subset PSL_2(\mathbb{C})$ is called a *Kleinian group*. Discreteness of Γ is equivalent to the action on $\mathbb{H}^3$ being properly discontinuous. Proper discontinuity easily implies Γ is torsion free if and only if it contains no elliptic elements. Γ is said to be *elementary* if it contains an Abelian subgroup of finite index, and *non-elementary* otherwise.

For the remainder of this section, let Γ represent a torsion free Kleinian group. We let $M_\Gamma = \mathbb{H}^3/\Gamma$ denote the quotient *hyperbolic 3-manifold* (with its induced metric), and let

$$p : \mathbb{H}^3 \to M_\Gamma$$

denote the projection. Note that $(\mathbb{H}^3, p)$ is the universal covering of M_Γ, and Γ is the group of covering transformations, so that $\pi_1(M_\Gamma) \cong \Gamma$. When we wish to refer to this isomorphism explicitly, we will write it as

$$\psi : \pi_1(M_\Gamma) \to \Gamma.$$

We say that Γ has *finite co-volume* (resp. is *co-compact*) if M_Γ has finite total volume (resp. is compact). A *cusp* of M_Γ is a subset of M_Γ isometric to a set of the form B/Γ_P where

$$B = \{(x, y, t) \in \mathbb{H}^3 \,|\, t > 1\}$$

and $\Gamma_P \cong \mathbb{Z} \oplus \mathbb{Z}$ is a subgroup of $PSL_2(\mathbb{C})$ consisting entirely of parabolics which stabilizes B. It can be shown (see [3] for example) that when Γ has finite co-volume, M_Γ is the interior of a compact manifold with toroidal boundary, and that every boundary component of that compact manifold has a product neighborhood whose intersection with M_Γ is a cusp of M_Γ.

Remark 1.1.2. When the interior of a compact manifold M is hyperbolic, we will simply say that M is hyperbolic. In fact, when it is convenient (and no confusion will arise) we will refer to a compact manifold and its interior by the same name.

Let γ be a free homotopy class of essential loops in M_Γ. We say that γ is hyperbolic (resp. parabolic) if $\psi(\gamma')$ is hyperbolic (resp. parabolic) where

$\gamma' \in \pi_1(M_\Gamma)$ is a representative of the conjugacy class determined by γ. If γ is hyperbolic, then there exists a unique, length minimizing, geodesic representative for γ given by $Ax(\psi(\gamma'))/ < \psi(\gamma') >$, where $Ax(\psi(\gamma'))$ is the axis for $\psi(\gamma')$. If γ is parabolic, then there exists a sequence of loops γ_n representing γ such that the length of γ_n goes to 0 as n goes to infinity.

As a homotopy class of loops is peripheral if and only if it has representatives lying entirely in a cusp of M_Γ, the following is straightforward.

Theorem 1.1.2. *Let M_Γ be a finite volume hyperbolic 3-manifold. A homotopy class of essential loops in M_Γ is peripheral if and only if it is parabolic.*

Notation. It is common to blur the distinction between a particular cusp and the end of the manifold corresponding to that cusp. We will follow this convention, referring to both objects as cusps. The context will make it clear which we are referring to.

Fuchsian groups

Given any subgroup $G \subset PSL_2(\mathbb{C})$ and any (geometric) circle $\mathcal{C} \subset \widehat{\mathbb{C}}$ (i.e. circle or line in $\mathbb{C}$), define

$$Stab_G(\mathcal{C}) = \{g \in G \,|\, g(\mathcal{C}) = \mathcal{C},\, g \text{ preserves the components of } \widehat{\mathbb{C}} \setminus \mathcal{C}\}$$

A *Fuchsian group* is defined to be a discrete subgroup of $Stab_{PSL_2(\mathbb{C})}(\mathcal{C})$, for some circle $\mathcal{C}$. For any circle $\mathcal{C} \subset \widehat{\mathbb{C}}$ there exists $g \in PSL_2(\mathbb{C})$ such that $g(\mathcal{C}) = \widehat{\mathbb{R}}$. It follows that $gStab_{PSL_2(\mathbb{C})}(\mathcal{C})g^{-1} = PSL_2(\mathbb{R})$. Thus, a Fuchsian group is a Kleinian group conjugate into $PSL_2(\mathbb{R})$ by an element $g \in PSL_2(\mathbb{C})$.

Any circle $\mathcal{C}$ on $\widehat{\mathbb{C}}$ is the boundary of a hyperbolic plane $P_{\mathcal{C}} \cong \mathbb{H}^2$ embedded totally geodesically in $\mathbb{H}^3$ and conversely. In the notation of the previous paragraph, we have $P_{\mathcal{C}} = g^{-1}(\mathbb{H}^2)$, where we view $\mathbb{H}^2 \subset \mathbb{H}^3$ as in Section 1.1.2. If Γ is a torsion free Fuchsian group stabilizing $\mathcal{C}$ (hence $P_{\mathcal{C}}$), $S_\Gamma = P_{\mathcal{C}}/\Gamma$ is a *hyperbolic surface*, and $\pi_1(S_\Gamma) \cong \Gamma$. Γ has *finite co-area* (resp. is *co-compact*) if S_Γ has finite total area (resp. is compact).

Suppose now that Γ is a finite co-volume torsion free Kleinian group such that there exists a circle $\mathcal{C} \subset \widehat{\mathbb{C}}$ for which $\Gamma' = Stab_\Gamma(\mathcal{C})$ has finite co-area. This induces a proper, totally geodesic immersion

$$S_{\Gamma'} \looparrowright M_\Gamma$$

It is immediate that any such surface is essential. Since the only (complete) totally geodesic surfaces in $\mathbb{H}^3$ are hyperbolic planes, any proper, totally geodesic immersion of an orientable, hyperbolic surface into M_Γ factors through an immersion of this type. That is, if $f : F \looparrowright M_\Gamma$ is a proper, totally geodesic immersion of an orientable, hyperbolic surface, then with the notation above, we have that $\psi \circ f_*(\pi_1(F)) \subset Stab_\Gamma(\mathcal{C})$ for some circle $\mathcal{C} \subset \widehat{\mathbb{C}}$.

We will be primarily interested in closed, totally geodesic surfaces. The classification of isometries of $\mathbb{H}^n$, $n = 2, 3$, easily implies

Theorem 1.1.3. *If F is a closed, totally geodesic surface in a finite volume hyperbolic 3-manifold M_Γ, then every free homotopy class of essential loops in F is hyperbolic.*

Theorem 1.1.2 and Theorem 1.1.3 together imply

Corollary 1.1.4. *If F is a closed, totally geodesic surface in a finite volume hyperbolic 3-manifold M_Γ, then there are no free homotopy classes of essential loops in F that are peripheral.*

Remark 1.1.3. The surfaces $S_{\Gamma'}$ corresponding to Fuchsian subgroups $Stab_{\Gamma}(\mathcal{C}) = \Gamma' \subset \Gamma$ are orientable, although it may be that the map $S_{\Gamma'} \looparrowright M_{\Gamma}$ factors through a covering of a non-orientable totally geodesic surface $\Sigma \looparrowright M_{\Gamma}$. This will happen if and only if there is a $g \in \Gamma$ such that $g(\mathcal{C}) = \mathcal{C}$ and g *exchanges* the components of $\widehat{\mathbb{C}} \setminus \mathcal{C}$.

Chapter 2

Virtually fibering Dehn fillings

Let M be a compact orientable 3-manifold with boundary consisting of a (possibly empty) union of tori. M is said to be *fibered* if it can be given the structure of a surface bundle over the circle. If M is finitely covered by a fibered manifold, then M is said to be *virtually fibered*. Since fibering persists in finite covers, the property of being virtually fibered is an invariant of the commensurability class.

If M is a Seifert fibered manifold, it is known (see [12] for example) that M is virtually fibered if and only if either the rational Euler number of the Seifert fibration or the orbifold Euler characteristic of the base is zero. When M is hyperbolic, the situation is considerably more complicated.

Although Thurston's Question 1.0.1 is conjectured to have an affirmative answer, finding examples of hyperbolic 3-manifolds which are not fibered, but are virtually fibered, is a hard problem in general. An elementary construction is to take M to be the union of two twisted I-bundles over a surface Σ glued together along their ∂I-bundles. There is an obvious 2-fold cover which is fibered (each of the I-bundles is covered by a product I-bundle and these can be glued together to obtain a bundle cover of M). For $\chi(\Sigma) < 0$

the gluing map can be chosen so that M is hyperbolic and so that M is not fibered (see [32] or [12]). In [12] Gabai gives examples of nonfibered virtually fibered link complements (with 2 or more components), and shows that they are not obtained by the above construction. In the same paper, Gabai also describes how to construct closed Haken manifolds with the same properties. In [32], Reid gives examples of non-Haken (in particular, nonfibered) virtually fibered closed hyperbolic 3-manifolds. Because the manifolds are non-Haken, they cannot be the union of two twisted I-bundles.

Let W be the Whitehead link, $\partial_0 X(W)$ one component of $\partial X(W)$, and $X(W)(p,q)$ the result of (p,q)-Dehn filling on $\partial_0 X(W)$ (with respect to the basis m and l shown in Figure 2.2). Our main result is

Theorem 2.1.2 $X(W)(p,q)$ *is virtually fibered.*

Combining this with the following result of Hodgson, Meyerhoff, and Weeks [16], we see that quite often the Dehn fillings produce manifolds which are not fibered but are virtually fibered.

Theorem 2.0.5. $X(W)(p,q)$ *is fibered if and only if* $q = \pm 1$ *or* $q = 0$.

□

In particular, this provides what seems to be the first known examples of nonfibered virtually fibered hyperbolic knot exteriors in S^3.

Let K_k denote the k-twist knot, $k \in \mathbb{Z}$. K_5 is shown in Figure 2.1

Corollary 2.5.1 *$X(K_k)$ is virtually fibered for all $k \in \mathbb{Z}$. Moreover, when $k > 2$ or $k < -3$, $X(K_k)$ is hyperbolic and not fibered.*

Remark 2.0.4. Since $H_1(X(K_k); \mathbb{Z}) \cong \mathbb{Z}$, it is easy to see that these are not the union of two twisted I-bundles glued along their ∂I-bundles.

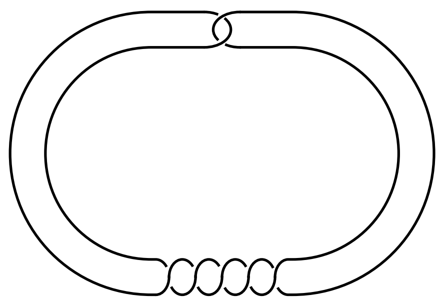

Figure 2.1: The knot K_5.

We will prove Theorem 2.1.2 by proving the following stronger result (see section 2.1 for the proof that Theorem 2.4.1 implies Theorem 2.1.2).

Let $C(p, s)$ denote the p-component chain link with s left handed half-twists (when s is negative, we will interpret this as meaning $-s$ right handed half twists). $C(6, -4)$ is shown in Figure 2.3

Theorem 2.4.1 *$X(C(p, s))$ is virtually fibered if and only if $(p, s) \neq (2, -1)$. Moreover, $X(C(p, s))$ is fibered if and only if $-2 \leq -s \leq p + 2$ and $(p, s) \neq (2, -1)$.*

The remainder of this chapter is organized as follows: Section 2.1 contains a proof of a result of Neumann and Reid [27] which determines a commensurability relationship between pairs of chain link exteriors needed to prove Theorem 2.4.1. In Section 2.2 we recall the notion of the Murasugi sum of oriented surfaces in S^3 and a theorem of Gabai [8] which allows us to detect fibered link complements. Section 2.3 contains standard facts concerning the Thurston norm [40], which allows us to show that certain link exteriors are not fibered. Theorem 2.4.1 is proved in Section 2.4. In Section 2.5 we briefly discuss applications of Theorem 2.4.1 and Theorem 2.1.2. Finally, in Section 2.6, we investigate the topology of the virtual fibrations, and show that all the manifolds we are considering are covered by punctured torus bundles with the obvious exceptions (see Theorem 2.6.1).

2.1 Commensurability of certain chain links

Given a projection of $C(p, s)$ in which the components are arranged in a circle, as in Figure 2.3, we will refer to a *clasp* as a pair of crossings where the two ends of "adjacent" components (or of the same component in the case $p = 1$) are linked. We will refer to this type of crossing as a *clasp crossing*, while a crossing which arises from twisting we refer to as a *twist crossing*.

The following theorem of Neumann and Reid [27] will be important in our work. We include its proof for completeness.

Theorem 2.1.1. *Let $p \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$. If $p + 2s \neq 0$, then $X(C(p, s))$ and $X(C(\pm(p + 2s), \mp(p + s)))$ are commensurable. Moreover, if s is even and $gcd(p, \frac{s}{2}) = 1$, then $X(C(p, s))$ is a p-fold cover of $X(W)(p, \frac{s}{2})$.*

Remark 2.1.1. The sign in $X(C(\pm(p+2s), \mp(p+s)))$ is whichever makes the first entry positive.

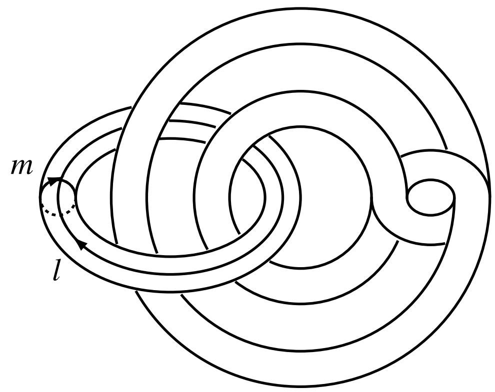

Figure 2.2: $X(W)$ and preferred basis m,l for $H_1(\partial_0 X(W))$.

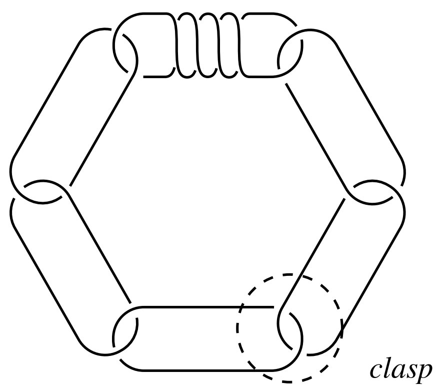

Figure 2.3: $C(6, -4)$ with a clasp indicated.

Proof. We will prove that $X(C(p, s))$ and $X(C(\pm(p+2s), \mp(p+s)))$ cover the same orbifold, hence are commensurable by Lemma 1.1.1.

The proof will make use of certain symmetries of $X(C(p, s))$ (in fact they are symmetries of $(S^3, C(p, s))$). There is a counterclockwise rotation taking each component to the next, which we call α, and a rotation of order two about the circular axis, which we will call β (see Figure 2.4). Let

$$G(p, s) = < \alpha, \beta > .$$

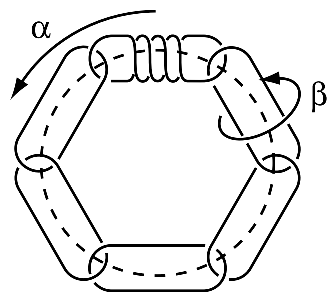

Figure 2.4: The symmetries α and β.

Explicitly, we may embed $C(p,s)$ in S^3 so that

$$\alpha(z_1,z_2) = (e^{\frac{2\pi i}{p}}z_1, e^{\frac{-s\pi i}{p}}z_2)$$

$$\beta(z_1,z_2) = (z_1,-z_2)$$

Let $X(W_0)$ be the orbifold 2-fold covered by $X(W)$ pictured in Figure 2.5. Let $\partial_0 X(W_0)$ be the torus component of $\partial X(W_0)$ (which is covered by $\partial_0 X(W)$) and $X(W_0)(p,s)$ the orbifold obtained from (p,s)-Dehn filling on $\partial_0 X(W_0)$ (with respect to m' and l' shown in Figure 2.5). In the case $p > 0$, this is the quotient of $X(C(p,s))$ by the group $G(p,s)$. To see this, let $V_1 \subset S^3$ be a solid torus containing $C(p,s)$ that is invariant under $G(p,s)$ (The situation for $C(3,2)$ is shown in Figure 2.6), and let $V_2 = \overline{S^3 \setminus V_1}$ be the complementary solid torus. We note that

$$X(C(p,s))/G(p,s) = (V_1 \setminus N(C(p,s)))/G(p,s) \cup V_2/G(p,s)$$

One easily verifies that $X(W_0) \cong (V_1 \setminus N(C(p,s)))/G(p,s)$ and that $V_2/G(p,s)$ is an orbifold solid torus with core curve having local group $\mathbb{Z}/d\mathbb{Z}$, where $d = |stab(0,z)| = gcd(p,s)$ for any $(0,z) \in S^3$. Therefore, $X(C(p,s))/G(p,s)$ is some Dehn filling on $\partial_0 X(W_0)$. One can verify that the $(p/d, s/d)$ curve bounds an (orbifold) disk, and hence $X(C(p,s))/G(p,s) \cong X(W_0)(p,s)$, as required. It will therefore suffice to prove that $X(W_0)(p,s)$ and $X(W_0)(p+2s, -p-s)$ cover the same orbifold, since $X(W_0)(p,s) = X(W_0)(-p,-s)$.

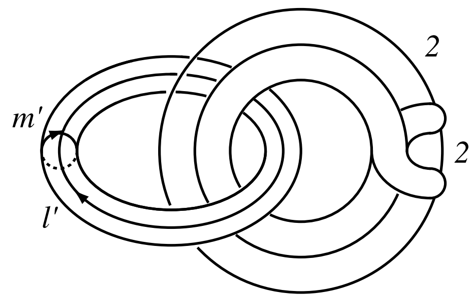

Figure 2.5: $X(W_0)$ and preferred basis m',l' .

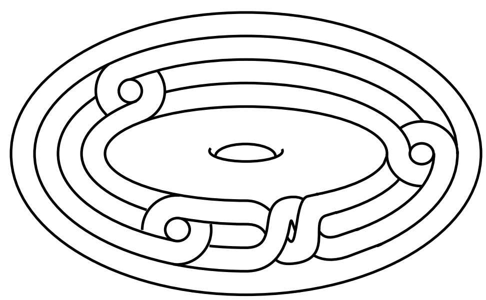

Figure 2.6: Solid torus containing $C(3,2)$.

We note that if s is even and $gcd(p, \frac{s}{2}) = 1$, $|G(p,s) :< \alpha > | = 2$ and $< \alpha >$ acts freely on $X(C(p,s))$. This fact, and a similar argument to that given in the previous paragraph shows that $X(C(p,s))$ is a p-fold cover of $X(W)(p, \frac{s}{2})$. This proves the second part of Theorem 2.1.1.

We let $X(W_1)$ be the orbifold shown in Figure 2.7, which is covered by $X(W_0)$ in a 2 to 1 fashion, and let $\partial_0 X(W_1) \subset \partial X(W_1)$ be the pillow case covered by $\partial_0 X(W_0)$. Notice that $X(W_1)$ is the quotient of $X(W_0)$ by an order two rotation τ. As τ extends to a homeomorphism (which we will also call τ) of any Dehn filling on $\partial_0 X(W_0)$, we let $X(W_1)(p, s)$ be the quotient of $X(W_0)(p, s)$ by τ.

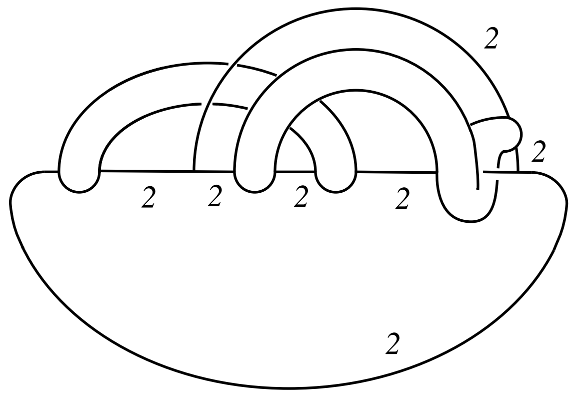

Figure 2.7: $X(W_1)$.

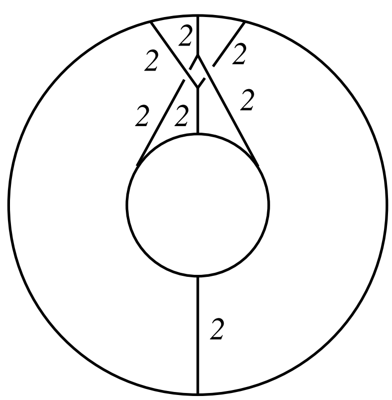

Figure 2.8: A more symmetric picture of $X(W_1)$.

We can redraw $X(W_1)$ as in Figure 2.8. Here we see that $X(W_1)$ admits an order 2 rotational symmetry ρ about a vertical axis. It can be checked that ρ restricted to $\partial_0 X(W_1)$ lifts to a homeomorphism $\widetilde{\rho}$ of $\partial_0 X(W_0)$ whose induced map on $H_1(\partial_0 X(W_0);\mathbb{Z})$ with respect to m' and l' is given by the matrix:

$$\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Therefore, given $X(W_0)(p,s)$ and $X(W_0)(p+2s,-p-s)$, we can extend $\widetilde{\rho}$ (which is a map defined only on $\partial_0 X(W_0)$), to a map from the glued in solid torus of $X(W_0)(p,s)$ to that of $X(W_0)(p+2s,-p-s)$. In fact, this can be done so that this extended map commutes with τ. Hence, we can define an (orbifold) homeomorphism of $X(W_1)(p,s)$ onto $X(W_1)(p+2s,-p-s)$ which extends ρ. Thus $X(W_0)(p,s)$ and $X(W_0)(p+2s,-p-s)$ cover the same orbifold, as required. □

We now prove Theorem 2.1.2 assuming Theorem 2.4.1.

Theorem 2.1.2. *$X(W)(p,q)$ is virtually fibered.*

Proof. We need to show that for any pair of co-prime integers p and q, $X(W)(p,q)$ is virtually fibered. Without loss of generality, we may assume that $p \geq 0$.

If $p > 0$, Theorem 2.1.1 states that $X(W)(p,q)$ is finitely covered by $X(C(p,2q))$. Since $(p,2q) \neq (2,-1)$, Theorem 2.4.1 implies $X(C(p,2q))$ is virtually fibered, hence so is $X(W)(p,q)$. If $p = 0$, we must have $q = 1$. The exterior of the Whitehead link is itself fibered [36], and we are filling along the boundary of a fiber. The fibering clearly extends over the filled solid torus and therefore $X(W)(0,1)$ is fibered. □

2.2 Murasugi sum

Here we recall the notion of Murasugi sum (see [11] or [25]) and its applications to constructing fibered links.

Definition 2.2.1. We say that the oriented surface R in S^3 with boundary L is the *Murasugi sum* of the two oriented surfaces R_1 and R_2 with boundaries L_1 and L_2 if there exists a 2-sphere S in S^3 bounding the balls B_1 and B_2 with $R_i \subset B_i$ for $i = 1, 2$, such that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = D$ where D is a $2n$-sided disk contained in S (see Figure 2.9).

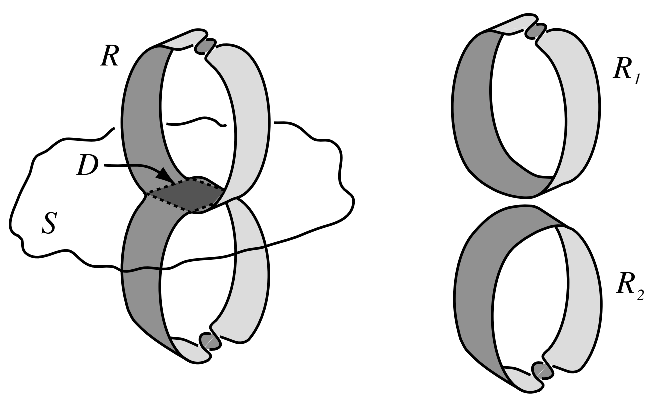

Figure 2.9: Murasugi summing to obtain a fiber for the figure 8 knot exterior.

The main result concerning the Murasugi sum is the following, due to Gabai [8].

Theorem 2.2.1. *Let $R \subset S^3$, with $L = \partial R$, be a Murasugi sum of oriented surfaces $R_i \subset S^3$, with $L_i = \partial R_i$, for $i = 1, 2$. Then $X(L)$ is fibered with fiber $R \setminus N(L)$ if and only if $X(L_i)$ is fibered with fiber $R_i \setminus N(L_i)$ for $i = 1, 2$.*

□

Remark 2.2.1. We will actually only use the "if" direction of Theorem 2.2.1. This direction that was proven partially by Murasugi [26] and completely by Stallings [38].

Remark 2.2.2. As an abuse of notation, we will refer to the fiber as R rather than $R \setminus N(L)$. We will also often shorten the statement "$X(\partial R)$ is fibered with fiber R" to "R is a fiber".

The following example of a fibered link exterior in S^3 will be basic to our construction.

Let H_l and H_r be the left and right handed Hopf bands shown in Figure 2.10 on the left and right respectively. This is an annulus spanning the left and right handed Hopf links, $L_l = \partial H_l$ and $L_r = \partial H_r$. H_l and H_r are both fibers since $X(L_l) \cong X(L_r) \cong T^2 \times I$ and H_r and H_l are annuli of the form $S^1 \times I$ (see [36], for example). Note that by Theorem 2.2.1, the Murasugi sum of the two Hopf bands pictured in Figure 2.9 is a fiber in the fibering of the exterior of the figure 8 knot.

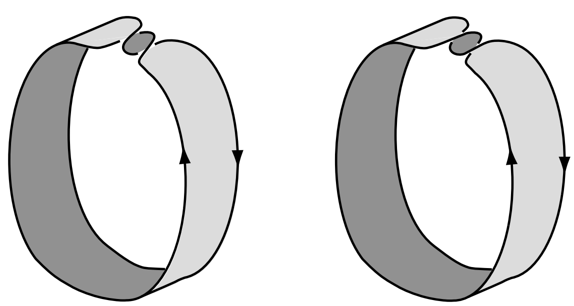

Figure 2.10: Left and right Hopf bands.

2.3 Thurston norm

In this section we recall the definition of the Thurston norm and a few of its properties. Throughout, M will denote a compact, irreducible, orientable 3-manifold with (possibly empty) toroidal boundary.

Definition 2.3.1. Given a compact, orientable surface F, we can express F as

$$F = F_+ \amalg F_-$$

where F_+ is the union of components of F with positive Euler characteristic and F_- is the union of all other components. Define

$$\chi_-(F) = -\chi(F_-)$$

where χ denotes Euler characteristic.

Since $H_2(M, \partial M; \mathbb{Z})$ is torsion free, we may view $H_2(M, \partial M; \mathbb{Z}) \subset H_2(M, \partial M; \mathbb{R})$ as the set of lattice points. Given $\phi \in H_2(M, \partial M; \mathbb{Z})$ define

$$||\phi||_T = \min_{[F]=\phi} \chi_-(F).$$

Here, the minimum is taken over all compact, oriented surfaces representing the class ϕ. The number $||\phi||_T$ is called the *Thurston norm* of ϕ.

A few elementary properties of $|| \, ||_T$ are described by the following theorem of Thurston [40].

Theorem 2.3.1. *$|| \, ||_T$ has a unique extension to a pseudo-norm*

$$|| \, ||_T : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$$

If no class of $H_2(M, \partial M; \mathbb{Z})$ is represented by a surface of non-negative Euler characteristic (e.g. if M is hyperbolic), then $|| \, ||_T$ is a norm.

Note that any norm $|| \, ||$ on a vector space V is completely determined by its unit ball

$$B = \{v \in V \mid ||v|| \leq 1\}$$

So, to describe $|| \, ||$, it suffices to describe B.

Because $||\phi||_T \in \mathbb{Z}$ for any $\phi \in H_2(M, \partial M; \mathbb{Z})$, there are special restrictions on the structure of the *Thurston norm ball*, B_T (the norm ball for $|| \, ||_T$). These are illustrated in the following theorem of Thurston [40].

Theorem 2.3.2. *When* $|| \, ||_T$ *is a norm, there are a finite number of lattice points* $\pm\beta_1, ..., \pm\beta_n \in H^2(M, \partial M; \mathbb{R})$ *such that*

$$B_T = \{\phi \in H_2(M, \partial M; \mathbb{R}) \mid |\beta_j \cdot \phi| \leq 1 \, \forall j = 1, ..., n\}$$

In particular, B_T *is a compact, convex polyhedron which is symmetric about the origin.*

We will use the Thurston norm to prove that certain manifolds do not fiber. The key theorem we will need is Theorem 2.3.3, again due to Thurston [40].

Given a face F of B_T, set $C(F)$ to be the cone on the interior of F, that is

$$C(F) = \mathbb{R}_{\geq 0} \cdot int(F).$$

Theorem 2.3.3. *Suppose* $|| \, ||_T$ *is a norm. There is a (possibly empty) set of faces* $F_1, ..., F_k$ *of* B_T *such that* $\phi \in H_2(M, \partial M; \mathbb{Z})$ *is represented by a fiber if and only if* $\phi \neq 0$ *and*

$$\phi \in \cup_{j=1}^{k} C(F_j)$$

Remark 2.3.1. The faces, F_j, described in this theorem are referred to as *fiber faces.*

One can use this theorem to prove that M does not fiber by checking only a finite number of homology classes (one in the cone of each open face of B_T) and showing that these classes are not represented by fibers. To see this strategy carried out, see e.g. [12] (or section 2.4).

2.4 Virtually fibering chain links

This section is devoted to proving the following

Theorem 2.4.1. *$X(C(p,s))$ is virtually fibered if and only if $(p,s) \neq (2,-1)$. Moreover, $X(C(p,s))$ is fibered if and only if $-2 \leq -s \leq p+2$ and $(p,s) \neq (2,-1)$.*

The proof will be split into a sequence of lemmas.

Lemma 2.4.2. *$X(C(p,s))$ is fibered if $-2 \leq -s \leq p+2$ and $(p,s) \neq (2,-1)$.*

Proof. When $p = 1$ we have $-3 \leq s \leq 2$. In these cases, $C(p,s)$ is the figure 8 knot, trefoil, and unknot, each of which has fibered exterior.

We may now suppose $p \geq 2$. First consider the case that $s \leq 0$. Since $0 \leq -s \leq p+2$ (and $(p,s) \neq (2,-1)$), we note that $C(p,s)$ has a projection given by Figure 2.11(a) ($s \leq -2$) or Figure 2.11(b) ($-2 < s \leq 0$). To see this we proceed as follows.

In the case $s \leq -2$, we can remove a twist crossing by changing a pair of clasp crossings (see Figure 2.12). We do this for all but 2 of the $-s$ crossings,

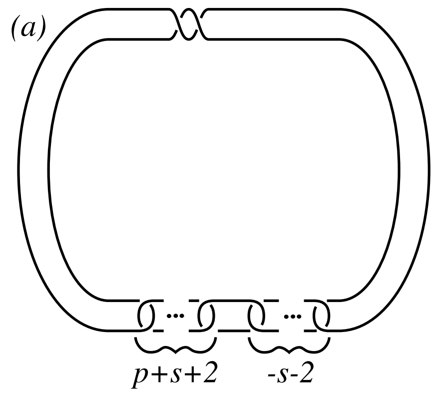

Figure 2.11: Projection of $C(p,s)$ for $s \leq -2$ (a) and $-2 < s \leq 0$ (b).

which is possible since $0 \leq -s-2 \leq p$. When $s = -1$, we can similarly change the single twist crossing and add a left handed twist crossing by changing three pairs of clasp crossings (this is where we need $(p,s) \neq (2,-1)$). When $s = 0$, we can add two left handed twist crossings by changing two pairs of clasp crossings.

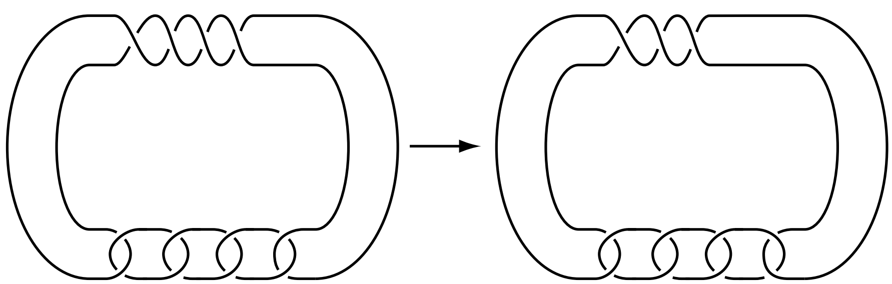

Figure 2.12: Removing a twist crossing by changing a pair of clasp crossings.

Now orient the link and perform Seifert's algorithm to obtain the Seifert surface $R(p,s)$ shown in Figure 2.13(a)($s \leq -2$) and Figure 2.13(b) ($-2 < s \leq 0$). We claim that this surface is a fiber in a fibering of $X(C(p,s))$.

From Figure 2.13 it is clear that for $s \leq -2$, $R(p,s)$ is the result of a

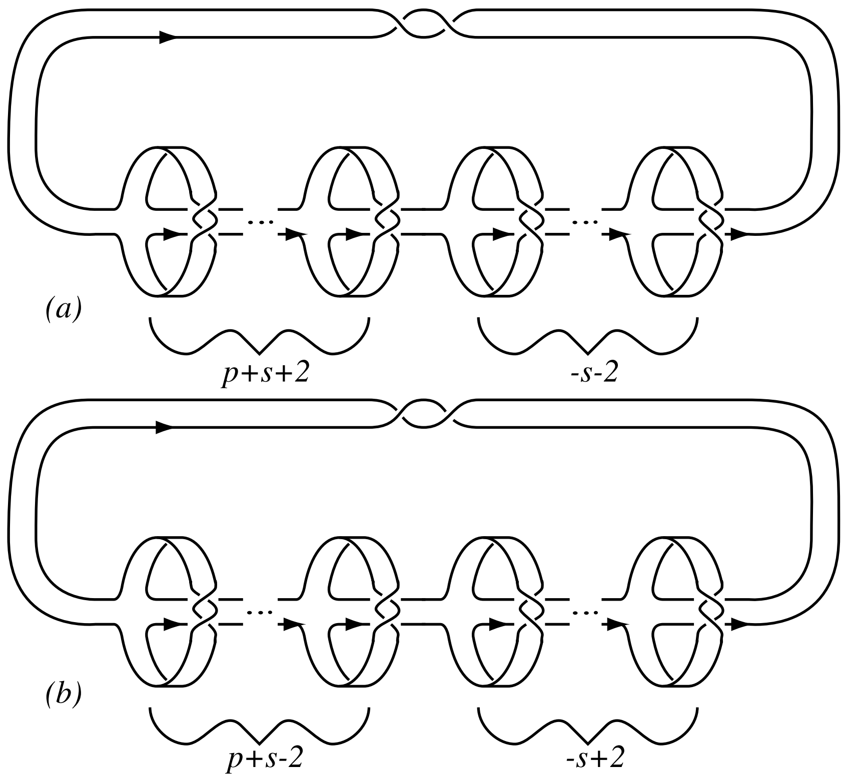

Figure 2.13: Seifert surface $R(p, s)$ for $s \leq -2$ (a) and $-2 < s$ (b).

Murasugi sum of $p + s + 2$ copies of H_l and $-s - 2$ copies of H_r to a single copy of H_l. Similarly, for $-2 < s \leq 0$, $R(p, s)$ is the result of a Murasugi sum of $p + s - 2$ copies of H_l and $-s + 2$ copies of H_r to a copy of H_r. Applying Theorem 2.2.1 and induction, we see that $R(p, s)$ is a fiber.

All that remains is the case that $0 < s \leq 2$. We could proceed as above, but rather we notice the following homeomorphism (see [27]). As above, we can add p left handed twist crossings by changing all the clasp crossings. Reflecting in the projection sphere for the diagram changes all the crossings, giving our standard projection of $C(p, -p - s)$. This reflection thus defines a homeomorphism $X(C(p, s)) \cong X(C(p, -p - s))$. Since $0 < s \leq 2$, we see that

$0 < -(-p-s) \leq p+2$, and by the first case $X(C(p,-p-s))$ is fibered, so $X(C(p,s))$ is. □

Lemma 2.4.3. *$X(C(p,s))$ is not fibered if $2 < s$ or $p+2 < -s$.*

Proof. Note that as in the proof of Lemma 2.4.2 when $p+2 < -s$, we may remove p twist crossings by changing all the clasp crossings. Again, reflection in the projection sphere gives an orientation reversing homeomorphism between $X(C(p,s))$ and $X(C(p,-p-s))$. Since $p+2 < -s$, we see that $2 = -p+p+2 < -p-s$. It therefore suffices to prove the lemma in the case $2 < s$, which we now assume.

We begin with the case that $p = 1$. In this case, one may check that the leading coefficient of the Alexander polynomial of $C(1,s)$ is $\pm\lceil \frac{s}{2} \rceil$. Since the Alexander polynomial of a knot with fibered exterior is monic (see [36]), $X(C(1,s))$ is not fibered.

We now assume that $p > 1$. We will construct the Thurston norm ball, B_T, for $X(C(p,s))$ and show that there are no fiber faces.

First, we note that in [27], it is determined precisely which chain links are hyperbolic. Specifically, it is shown that $X(C(p_0,s_0))$ is hyperbolic if and only if

$$\{|p_0+s_0|, |s_0|\} \not\subset \{0,1,2\}$$

As we are assuming that $s > 2$, we see that $X(C(p,s))$ is hyperbolic, and so $|| \, ||_T$ is a norm by Theorem 2.3.1.

Next, we orient and label the components of $C(p,s)$ as shown in Figure 2.14. By excision and the exact homology sequence (see e.g. [14] or [40]), we have

$$H_2(X(C(p,s)),\partial X(C(p,s));\mathbb{R}) \cong H_2(S^3, C(p,s);\mathbb{R})$$
$$\cong H_1(C(p,s);\mathbb{R}) \cong H_1(\coprod_{j=1}^{p} S^1;\mathbb{R}) \cong \mathbb{R}^p.$$

We may therefore, view $\lambda_1, ..., \lambda_p$ as a basis for $H_2(X(C(p,s)),\partial X(C(p,s));\mathbb{R})$.

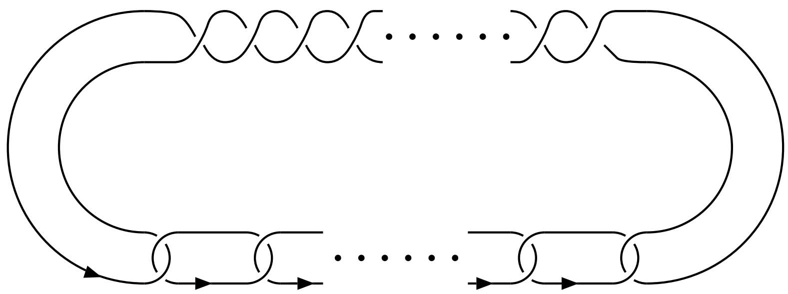

Figure 2.14: Orienting and labeling the components of $C(p,s)$.

Since $p > 1$, we see that for each $j = 1, ..., p$, λ_j is represented by a pair of pants, P_j (i.e. a sphere with 3 holes). Therefore, $||\lambda_j||_T \le \chi_-(P_j) = 1$ Since $||\,||_T$ is a norm and $||\lambda_j||_T$ is an integer

$$||\lambda_j||_T = 1.$$

Next, let $\epsilon = (\epsilon_1, ..., \epsilon_p) \in \{\pm 1\}^p$. We consider the element

$$\lambda_\epsilon = \sum_{j=1}^{p} \epsilon_j \lambda_j \in H_2(X(C(p,s)),\partial X(C(p,s)))$$

By the definition of a norm, we see that

$$||\lambda_\epsilon||_T = ||\sum_{j=1}^{p} \epsilon_j \lambda_j||_T \le \sum_{j=1}^{p} ||\epsilon_j \lambda_j||_T = \sum_{j=1}^{p} ||\lambda_j||_T = p$$

If this inequality is an equality (i.e. $||\lambda_\epsilon||_T = p$) then it follows by convexity and an elementary argument that the convex hull of $\epsilon_1\lambda_1, ..., \epsilon_p\lambda_p$ is a face, F_ϵ, of B_T.

Fixing $\epsilon \in \{\pm 1\}^p$, we see that the class λ_ϵ is the canonical class of the oriented link $C(p,s)$ with the j^{th} component oriented as $\epsilon_j\lambda_j$. We note that since $s > 2 > 0$, $C(p,s)$ is alternating. Therefore, by [25], [6], or [9], the surface $R = R(\lambda_\epsilon)$ obtained by applying Seifert's algorithm is Thurston norm minimizing. Near the clasps, this surface has one of the four forms shown in Figure 2.15, depending on the orientations of the components on both sides of the clasp. One can now easily check that the $\chi_-(R) = p$ (e.g. in $C(7, 2s)$, the surface $R(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7)$ deformation retracts onto the graph shown in Figure 2.16).

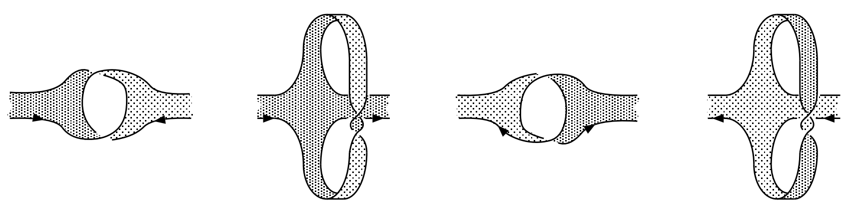

Figure 2.15: Four possible local pictures of $R(\lambda_\epsilon)$.

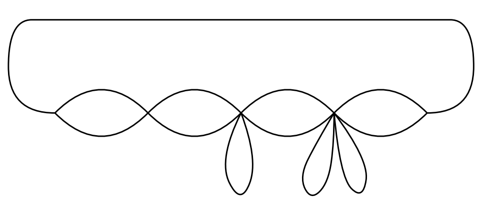

Figure 2.16: Deformation retract of $R(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7)$.

As noted above, we now have that the convex hull of $\epsilon_1\lambda_1, ..., \epsilon_p\lambda_p$ is a face $F_\epsilon \subset B_T$. It follows that B_T is exactly the convex hull of $\pm\lambda_1, ..., \pm\lambda_p$ (B_T is shown in Figure 2.17 for the case $p = 3$).

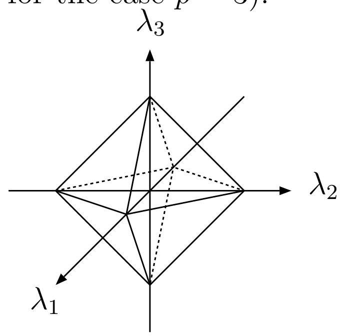

Figure 2.17: The ball B_T when $p = 3$

To complete the proof of the lemma, we need to show that no face F_ϵ is a fiber face (see section 2.3). By Theorem 2.3.3, it suffices to show that λ_ϵ is not represented by a fiber for any $\epsilon \in \{\pm 1\}^p$.

We compute the Alexander polynomial of the class λ_ϵ and show that it is not monic (as for knots, the Alexander polynomial of the class of a fiber is monic). We can compute this as in [36] from the linking form on the surface $R(\lambda_\epsilon)$. The leading coefficient is $\pm\frac{n+s}{2}$, where $n \geq 0$ is the number of clasps for which the orientation (from λ_ϵ) disagrees on opposite sides of the clasp (that is, in the first and third situation depicted in Figure 2.15). Note that we must have $n \equiv s \pmod{2}$, so that $\frac{n+s}{2}$ is an integer. Since $s > 2$, it follows that λ_ϵ is not represented by a fiber. $\square$

Lemma 2.4.4. *Given $p > 0$ and $s \leq 0$, $X(C(p,s))$ is virtually fibered if and only if $(p,s) \neq (2,-1)$.*

Proof. If $(p,s) = (2,-1)$, then $C(p,s)$ is the 2 component unlink. $X(C(p,s))$

is therefore a reducible orientable compact 3-manifold, and the only such manifolds which are virtually fibered are virtual sphere bundles, which are closed 3-manifolds. Hence $X(C(p,s))$ is not virtually fibered. This proves the "only if" part.

To prove the other implication we note that by Lemma 2.4.2 we may assume that $-s > p+2$. Since $p \geq 1$, this implies $s < -2$. Let

$$p' = -(p+2s)$$

$$s' = p + s$$

By our assumption, we have

$$s' = p + s < -2$$

and

$$p' = -p - 2s > s - 2s = -s > 2$$

This implies $p' > 3$ (in particular $(p', s') \neq (2, -1)$). We also have

$$p' = -(p+2s) = -(p+s) - s = -s' - s > -s'$$

The hypotheses of Lemma 2.4.2 are therefore satisfied by (p', s'), whence $X(C(p', s'))$ is fibered. Applying Theorem 2.1.1, we see that $X(C(p,s))$ and $X(C(p', s'))$ are commensurable. Thus $X(C(p,s))$ is virtually fibered. □

We now prove Theorem 2.4.1

Proof. By Lemma 2.4.2 and Lemma 2.4.4, we need only show that if $s > 2$, then $X(C(p,s))$ is virtually fibered. As in the proof of Lemma 2.4.3, there is an orientation reversing homeomorphism of $X(C(p,s))$ onto $X(C(p,-p-s))$. Since $s > 2$ and $p > 0$, we know that $-p-s < -1$. Therefore, by Lemma 2.4.4, $X(C(p,-s-p)) \cong X(C(p,s))$ is virtually fibered. □

2.5 Applications

2.5.1 Twist knots

As an immediate corollary of Theorem 2.4.1, we have

Corollary 2.5.1. *$X(K_k)$ is virtually fibered for all $k \in \mathbb{Z}$. Moreover, when $k > 2$ and $k < -3$, $X(K_k)$ is hyperbolic and not fibered.*

Remark 2.5.1. As noted in the proof of Lemma 2.4.2, for the cases $k = -3,-2,-1,0,1,2$ we have the figure 8, trefoil, unknot, unknot, trefoil, and figure 8, respectively, which are fibered. The figure 8 knot is the only one of these which is itself hyperbolic.

Proof. Applying Theorem 2.4.1 to $K_k = C(1,k)$, all that remains is to prove that $X(K_k)$ is hyperbolic. However, as was mentioned in the proof of Lemma 2.4.3, it is shown in [27] that $C(p,s)$ is hyperbolic if and only if

$$\{|p+s|, |s|\} \not\subset \{0,1,2\}.$$

It follows that K_k will be hyperbolic when

$$k > 1 \text{ or } k < -2.$$

□

2.5.2 The knot 7_4

As a consequence of Theorem 2.1.2 we record

Corollary 2.5.2. *The exterior of the knot 7_4 is not fibered but is virtually fibered.*

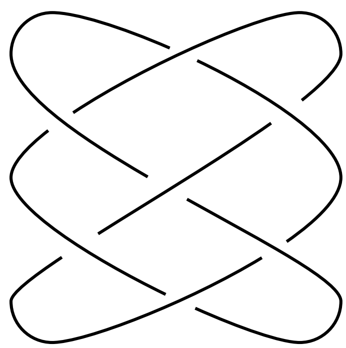

Figure 2.18: The knot 7_4.

Proof. The knot 7_4 from the knot tables [36] is pictured in Figure 2.18. Its Alexander polynomial is

$$4t^2 - 7t + 4$$

Hence 7_4 is not fibered.

To show that $X(7_4)$ is virtually fibered, we will show that it is commensurable with $X(W)(-3, 2)$. We use the same notation as in the proof of Theorem 2.1.1.

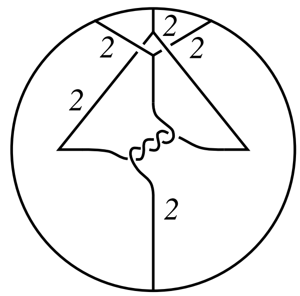

Figure 2.19: $X(W)(-3,2)$.

Since the $(-3,4)$ curve on $\partial_0 X(W_0)$ lifts to the $(-3,2)$ curve on $\partial_0 X(W)$, $X(W_0)(-3,4)$ is covered by $X(W)(-3,2)$. Let $X(W_1)(-3,4)$ be the quotient of $X(W_0)(-3,4)$ by τ. $X(W_1)(-3,4)$ is shown in Figure 2.19 (this a 3-ball with the singular locus as indicated). This can be redrawn as in Figure 2.20.

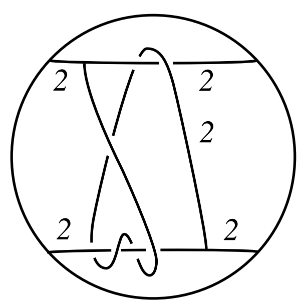

Figure 2.20: Another picture of $X(W)(-3,2)$.

Next, we see that $X(7_4)$ is a 4-fold cover of the orbifold in Figure 2.21 which we call $X(7_4)'$. It is the quotient by the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ symmetry group generated by order two rotations about vertical and horizontal axes.

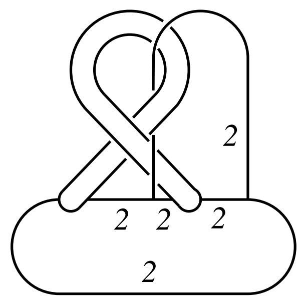

Figure 2.21: $X(7_4)'$; a quotient of $X(7_4)$.

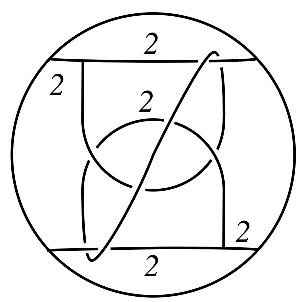

Figure 2.22: A more symmetric picture of $X(7_4)'$.

We can redraw $X(7_4)'$ as in Figure 2.22. From this picture one observes that $X(7_4)'$ admits an order two rotation about an axis which runs "perpendicular to the paper". This gives the quotient $X(7_4)''$ pictured in Figure 2.23.

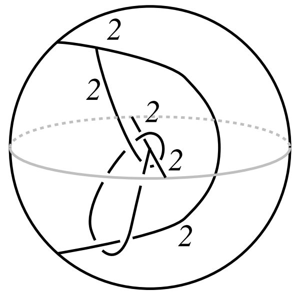

Figure 2.23: $X(7_4)''$; a quotient of $X(7_4)'$.

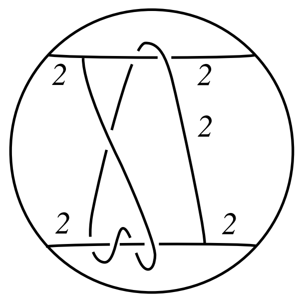

Figure 2.24: Another picture of $X(7_4)''$.

We redraw $X(7_4)''$ as in Figure 2.24, and observe that it is homeomorphic to $X(W_1)(-3, 4)$. □

2.5.3 Finitely generated intersection property

We briefly describe one final consequence of Theorem 2.1.2.

Recall that a group has the *finitely generated intersection property* (FGIP) if for every two finitely generated subgroups $H, K < G$, $H \cap K$ is also finitely generated. In [17] Jaco shows that if a compact 3-manifold M is virtually fibered with fiber F satisfying $\chi(F) < 0$, then $\pi_1(M)$ does not have FGIP.

Corollary 2.5.3. *$\pi_1(X(W)(p,q))$ has FGIP if and only if $(p,q) = (1,0)$.*

Proof. We again use the fact (see [27]) that it is known exactly which Dehn fillings are hyperbolic. There are exactly six surgeries which are fail to be hyperbolic. These are given by the slopes $(0,1)$,$(1,0)$,$(1,-1)$,$(2,-1)$,$(3,-1)$, and $(4,-1)$. The virtual fibers must have negative Euler characteristic.

Theorem 2.1.1 implies $X(W)(3,1)$ and $X(W)(4,1)$ have covers with at least 3 boundary components, so the virtual fiber must have $\chi(F) < 0$. Similarly, one can show that $X(W)(2,-1)$ has a cover with 3 boundary components. $X(W)(0,1)$ and $X(W)(1,-1)$ are punctured torus bundles. $X(W)(1,0)$ is a solid torus. □

Remark 2.5.2. This theorem also follows from Theorem 2.6.1.

2.6 The fibers

From the proof of Theorem 2.4.1, the virtual fibrations of $X(W)(p,q)$ and $X(C(p,s))$ are explicit in only "half" of the cases. In all the explicit cases, except when the manifold is a solid torus, the fibers are punctured-tori (with several punctures). This happens in general.

Theorem 2.6.1. *$X(W)(p,q)$ and $X(C(p,s))$ are covered by punctured-torus bundles (possibly with several punctures) except for the following cases.*

- *$X(W)(1,0)$, $X(C(1,0))$, and $X(C(1,-1))$ are solid tori and hence are not covered by punctured-torus bundles.*
- *$X(C(2,-1))$ is not virtually fibered.*

The proof of Theorem 2.6.1 requires a series of lemmas. Before we can state these, we will need the following definition.

Definition 2.6.1. If $\phi : F \to F$ is an automorphism of a compact surface F, and $p : \widetilde{F} \to F$ is a finite cover, then we say that ϕ lifts to $\widetilde{F}$ if there is an automorphism $\widetilde{\phi} : \widetilde{F} \to \widetilde{F}$ such that $p \circ \widetilde{\phi} = \phi \circ p$.

Note that ϕ lifts in this sense if and only if $\phi \circ p$ lifts to the covering $p : \widetilde{F} \to F$ in the usual sense (see e.g. [14]).

Our first lemma is the following.

Lemma 2.6.2. *Let $\phi : F \to F$ be an automorphism of a compact surface F and $\widetilde{F}$ a finite cover of F. There is a finite cover $\widetilde{F}_1$ of $\widetilde{F}$ such that ϕ lifts to $\widetilde{F}_1$.*

Proof. Let $\Gamma = \pi_1(F)$ and $\Gamma_0 = \pi_1(\widetilde{F})$, and we view $\Gamma_0 < \Gamma$. By passing to a further cover if necessary, we may assume that $\Gamma_0 \lhd \Gamma$. We will be done if we can find a finite index ϕ-invariant subgroup of Γ, which contains Γ_0. For then, by standard covering space theory, ϕ will lift to the cover corresponding to this subgroup.

Let

$$\rho : \Gamma \to G = \Gamma/\Gamma_0$$

be the canonical epimorphism, and note that $|G| < \infty$. Since Γ is finitely generated, there are only a finite number of homomorphisms from Γ onto G. Therefore, the the set $\{\rho \circ \phi^n\}_{n\in\mathbb{Z}}$ is finite. Let $N \subset \mathbb{Z}$ be a finite set such that

$$\{\rho \circ \phi^n\}_{n\in N} = \{\rho \circ \phi^n\}_{n\in\mathbb{Z}}$$

and define

$$\rho_N : \Gamma \to \prod_{n \in N} G$$

by

$$\rho_N(\gamma) = \{\rho \circ \phi^n(\gamma)\}_{n \in N}.$$

It follows that $ker(\rho_N) < \Gamma_0 < \Gamma$ is ϕ-invariant, and

$$[\Gamma : ker(\rho_N)] \leq |\prod_{n \in N} G| < \infty$$

□

Corollary 2.6.3. *Let M be a compact fibered 3-manifold with fiber F. For any cover $\widetilde{F}$ of F, there is a cover $\widetilde{F}_1$ of $\widetilde{F}$ and a cover $q : \widetilde{M} \to M$, such that $\widetilde{M}$ is fibered with fiber $\widetilde{F}_1$.*

Proof. Express M as the mapping torus of the monodromy $\phi : F \to F$. Let $q_1 : \widetilde{F}_1 \to F$ be the covering from Lemma 2.6.2 and $\widetilde{\phi} : \widetilde{F}_1 \to \widetilde{F}_1$ the corresponding lift. Construct $\widetilde{M}$ as the mapping torus of $\widetilde{\phi}$.

Since $\widetilde{\phi}$ is a lift of ϕ, the covering map $q_1 \times id_I : \widetilde{F}_1 \times I \to F \times I$ descends to a covering $q : \widetilde{M} \to M$. □

The following lemma is an easy consequence of standard covering space theory. We will use it to guarantee that certain covers of punctured tori are again punctured tori.

Lemma 2.6.4. *Let F be a compact surface (with $\partial F \neq \emptyset$) and $p : \widetilde{F} \to F$ a cover and suppose that every peripheral loop of F lifts to a loop in $\widetilde{F}$. Then p extends to a cover of the surfaces with their respective boundaries capped off with disks.*

Proof. Attach a disk to each component of $\partial\widetilde{F}$ and each component of ∂F. Since all peripheral loops lift to loops, it follows that for each boundary component $\partial_0\widetilde{F}$, $p|_{\partial_0\widetilde{F}}$ is a homeomorphism onto a boundary component of F. We may therefore extend this homeomorphism over the corresponding disks. Doing this for each boundary component, one easily verifies that the resulting map is a covering map. $\square$

We are now in a position to prove Theorem 2.6.1

Proof. We use the notation from the proofs of Theorem 2.1.1, Lemma 2.4.2, and Lemma 2.4.4. Since $X(W)(0,1)$ is a punctured torus bundle, we may assume that $p \neq 0$. Also, by Theorem 2.1.1, since $X(C(p,s))$ is a p-fold cover of $X(W)(p, \frac{s}{2})$, we need only show that that in the non-exceptional cases, $X(C(p,s))$ is covered by a punctured-torus bundle.

Case 1. $p > 1$, $-2 \leq -s \leq p+2$, $(p,q) \neq (2,-1)$.

In this case, the proof of Lemma 2.4.2 shows that the fibers $R(p,s)$ are punctured tori.

Case 2. $p > 0$, $-s > p+2$.

Set $p' = -(p+2s)$ and $s' = p+s$. From the proof of Theorem 2.1.1, we have regular orbifold covers

$$\pi_{(p,s)} : X(C(p,s)) \to X(W_0)(p,s)$$

$$\pi_{(p',s')} : X(C(p',s')) \to X(W_0)(p',s'),$$

$$\eta_{(p,s)} : X(W_0)(p,s) \to X(W_1)(p,s)$$

$$\eta_{(p',s')} : X(W_0)(p',s') \to X(W_1)(p',s')$$

and an orbifold homeomorphism

$$\psi : X(W_1)(p',s') \to X(W_1)(p,s).$$

If we let V and V' be the (orbifold) filling solid tori for $X(W_0)(p,s)$ and $X(W_0)(p',s')$, respectively, then $\psi \circ \eta_{(p',s')}(V') = \eta_{(p,s)}(V)$. Moreover, viewing $X(W_1)$ as a suborbifold of both $X(W_1)(p,s)$ and $X(W_1)(p',s')$, $\psi|_{X(W_1)}$ is the involution ρ from the proof of Theorem 2.1.1.

Now a careful inspection of the proof of Theorem 2.1.1 shows that the punctured-torus $R(p',s')$ from Case 1 may be chosen to be disjoint from $\pi^{-1}_{(p',s')}(V')$ (the core of $\pi^{-1}_{(p',s')}(V')$ is drawn in Figure 2.25). Let $i : R(p',s') \to X(C(p',s'))$ denote the inclusion.

Consider the map

$$f_{(p',s')} = \psi \circ \eta_{(p',s')} \circ \pi_{(p',s')} \circ i : R(p',s') \to X(W_1)(p,s)$$

and the subgroup

$$\Gamma_{(p,s)} = f^{-1}_{(p',s')*}((\eta_{(p,s)} \circ \pi_{(p,s)})_*(\pi_1(X(C(p,s))))) < \pi_1(R(p',s')).$$

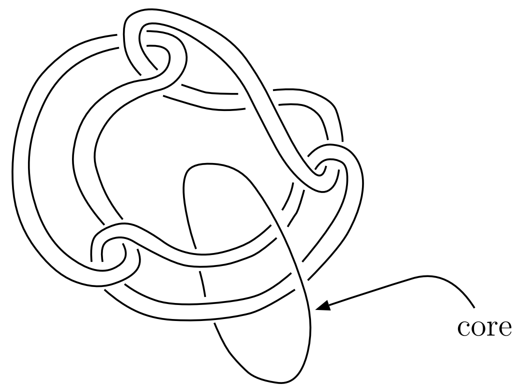

Figure 2.25: The core of $\pi^{-1}_{(3,-1)}(V)$ in $X(C(3,-1))$.

Claim. The cover $\nu_{(p,s)} : F(p,s) \to R(p',s')$ corresponding to $\Gamma_{(p,s)}$ is a punctured torus which extends to a cover of the surfaces with boundary components capped off with disks.

We postpone the proof of this claim for the moment, and see how it implies $X(C(p,s))$ is covered by a punctured-torus bundle. By Corollary 2.6.3, there is a cover $\epsilon : M(p,s) \to X(C(p',s'))$ such that the fiber $\widetilde{F}(p,s)$ is a punctured-torus covering $F(p,s)$. Here we are actually applying Corollary 2.6.3 to the manifold resulting from Dehn filling along each of the slopes of $R(p',s')$ on $\partial X(C(p',s')))$, then removing the preimages of the filling solid tori in the cover (this is to guarantee that the surfaces are punctured tori).

Now consider the cover $\widetilde{M}(p,s)$ of $X(C(p,s))$ corresponding to the intersection of $\pi_1(M(p,s))$ and $\pi_1(X(C(p,s)))$, both thought of as subgroups of $X(W_1)(p,s)$ (the former under the map $(\psi \circ \eta_{(p',s')} \circ \pi_{(p',s')} \circ \epsilon)_*$). Since $\pi_1(\widetilde{F}(p,s))$ is a subgroup of both $\pi_1(M(p,s))$ and $\pi_1(X(C(p,s)))$, $\widetilde{F}(p,s)$ lifts to this cover, and hence $\widetilde{M}(p,s)$ must be fibered with fiber $\widetilde{F}(p,s)$ (i.e. $\widetilde{M}(p,s)$ must be a finite cyclic cover of $M(p,s)$ obtained from the mapping torus of a power of the monodromy for $M(p,s)$). Therefore, $X(C(p,s))$ is covered by a punctured torus bundle.

Case 3. $p > 0$ and $s > 2$

In this case, (as in the proof of Theorem 2.4.1) we see that $X(C(p,s))$ is homeomorphic to $X(C(p,-s-p))$. Since $-(-s-p) = s+p > p$, $X(C(p,-s-p))$ falls into case 2, and is covered by a punctured-torus bundle, so $X(C(p,s))$ is also.

Case 4. Exceptional cases: $(p,s) = (1,0),(1,-1),(2,-1)$.

$X(C(1,0))$ and $X(C(1,-1))$ are solid tori and $X(C(2,-1))$ is not virtually fibered by Theorem 2.4.1.

All that remains is to prove the claim above.

Proof of claim. We wish to apply Lemma 2.6.4 to prove this claim. To do this, we need to know that the peripheral curves on $R(p',s')$ lift to $F(p,s)$. This will happen if the peripheral elements of $\pi_1(R(p',s'))$ are mapped into the subgroup $(\eta_{(p,s)} \circ \pi_{(p,s)})_*(\pi_1(X(C(p,s))))$ by $f_{(p',s')*}$.

Since $f_{(p,s)} = (\eta_{(p',s')} \circ \pi_{(p',s')}) \circ i_{(p',s')}$, we first determine the i_*-image of the the peripheral elements of $\pi_1(R(p',s'))$. Recall that in our situation $p' > 3$ (see the proof of Lemma 2.4.4). Therefore, each component $\partial_0 X(C(p',s')) \subset \partial X(C(p',s'))$ is the boundary of a neighborhood of an unknotted component $K_0 \subset C(p',s')$ in S^3. Fix such a component $\partial_0 X(C(p',s'))$ for the remainder of the proof. Let λ denote the slope of the disk in S^3 bounded by K_0, intersected with $\partial_0 X(C(p', s'))$, and let μ denote the meridian disk of a neighborhood K_0, (see Figure 2.26).

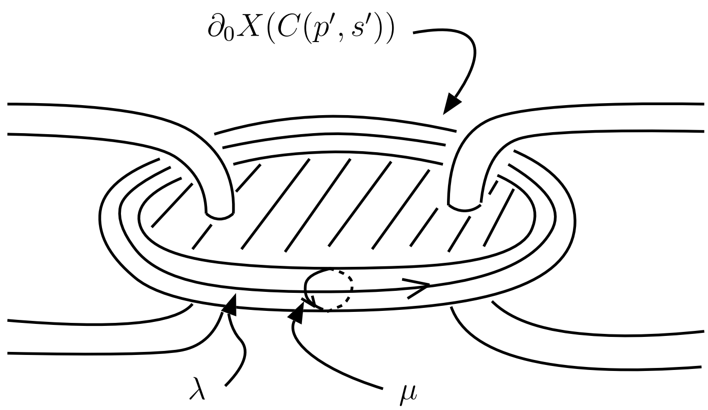

Figure 2.26: λ and μ on $\partial_0 X(C(p', s'))$.

We wish to determine the slopes of the surface $R(p', s')$ on $\partial_0 X(C(p', s'))$. From the proof of Lemma 2.4.2 there are four cases depending on the clasps at the two ends of K_0. We can draw these four cases as in Figure 2.27(a),(b),(c), and (d).

A computation shows that in cases (a) and (b), the slopes are $(\pm 2, 1)$ with respect to the ordered basis μ, λ. In cases (c) and (d), the the slopes are both $(0, 1)$. We refer to the loops on $\partial_0 X(C(p', s'))$ corresponding to the slopes $(2, 1)$, $(-2, 1)$, and $(0, 1)$ as γ_1, γ_2, and γ_3, respectively. When it is convenient, we will refer to elements in $\pi_1(X(C(p', s')))$ by the same names.

We are required to show that $(\eta_{(p',s')} \circ \pi_{(p',s')})_*(\gamma_i)$ is in the image of $(\eta_{(p,s)} \circ \pi_{(p,s)})_*(\pi_1(X(C(p, s))))$, for $i = 1, 2, 3$. This is equivalent to showing that $(\eta_{(p',s')} \circ \pi_{(p',s')})_*(\gamma_i)$ lift to loops in $X(C(p, s))$, for $i = 1, 2, 3$. To prove this, it is enough to show that $(\eta_{(p',s')} \circ \pi_{(p',s')})_*(\gamma_i)$ lift to loops in $X(W_0)(p, s)$,

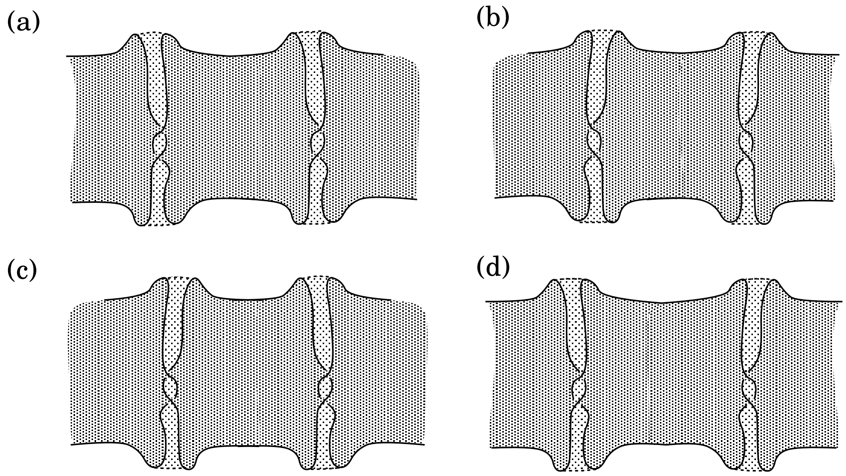

Figure 2.27: The four possibilities for $R(p', s')$ near $\partial_0 X(C(p', s'))$.

for $i = 1, 2, 3$. This is because $\partial X(W_0)(p, s)$ is a pillow case, and each of the boundary components of $X(C(p, s))$ is just a two-fold orbifold cover by a torus. Thus, every essential loop in $\partial X(W_0)(p, s)$ lifts to a loop.

Chasing through all the covers, one sees that the condition for a curve on $\partial_0 X(C(p', s'))$ to push down to a curve which lifts to $\partial X(W_0)(p, s)$ is that the mod 2 intersection number with λ is 0. Since each γ_i satisfies this property, the claim is established. This completes the proof of theorem. $\square$

Chapter 3

Compressing totally geodesic surfaces

Let M be a compact, orientable, irreducible 3-manifold with torus boundary $\partial M \cong T^2$. It was shown in [5] that if F is a closed, orientable, embedded, essential surface in M admitting no incompressible annulus with one boundary component contained in F and the other in ∂M, and F compresses in $M(\alpha)$ and $M(\beta)$, then $\Delta(\alpha, \beta) \leq 2$. This result was later improved to $\Delta(\alpha, \beta) \leq 1$, [42].

In this chapter we study the analog of the above result for immersed surfaces. When M is hyperbolic, a closed, immersed, totally geodesic surface is an essential surface having no incompressible annulus as above. These surfaces have good incompressibility properties, for they remain incompressible (and hence essential) in all but a finite number of Dehn fillings on one cusp (this can be deduced from [24] and [39], see also [2] for explicit bounds).

The hyperbolic manifold we will be primarily interested in is the exterior of the figure eight knot in S^3. We will refer to this manifold as M_8. Using arithmeticity of M_8, one can show that it contains infinitely many closed, immersed totally geodesic surfaces [23],[34].

The main result of this chapter is

Theorem 3.2.1 *Suppose* $4|p$ *and* $3 \nmid p$*. Then, for any* q *with* $gcd(p, q) = 1$ *there exists infinitely many non-commensurable, closed, orientable, immersed, totally geodesic surfaces in* M_8 *which compress in* $M_8(p, q)$*.*

Theorem 3.2.1 implies that there is no global bound on the number of fillings which one must omit to guarantee incompressibility of a closed totally geodesic surfaces. More precisely, we have

Corollary 3.0.5. *There are infinitely many Dehn fillings on* ∂M_8 *such that for each filling, some immersed, closed, totally geodesic surface compresses in the filled manifold.*

Since every surface in M_8 compresses in $M_8(1, 0)$, Theorem 3.2.1 implies the following result, indicating there is no analog in the immersed setting of [5] and [42].

Theorem 3.1.7 *There exists a compact, orientable, irreducible* 3*-manifold* M*, with torus boundary, having the following property. Given any positive integer,* n*, there exists a closed, orientable, immersed, essential surface* $F \looparrowright M$ *with no incompressible annulus joining* F *and* ∂M*, such that* F *compresses in* $M(\alpha)$ *and* $M(\beta)$ *and* $\Delta(\alpha, \beta) > n$*.*

Remark 3.0.1. We should note that although Theorem 3.1.7 guarantees the existence of surfaces which compress in fillings arbitrarily far apart, it does not guarantee that the surfaces compress in arbitrarily *many* of the filled manifolds.

Remark 3.0.2. The orientability of the surfaces is not an important part of either Theorem 3.2.1 or Theorem 3.1.7 since any immersed non-orientable surface may be replaced by an orientable one by taking the orientable double cover.

The rest of this chapter is organized as follows: Section 3.1 contains results from the theory of arithmetic hyperbolic manifolds needed to prove Theorem 3.2.1. We complete this section by proving Theorem 3.1.7, assuming Theorem 3.2.1. In Section 3.2, we prove Theorem 3.2.1 and describe some other examples obtainable using a similar construction. Finally, Section 3.3 contains some specific calculations from the proof of Theorem 3.2.1.

3.1 Arithmetic manifolds

We discuss background and results from the theory of arithmetic manifolds, applications to the figure eight knot, and we give a proof of Theorem 3.1.7. The results in this section (with the exception of Theorem 3.1.7) are known but some proofs have been included for the sake of completeness. For more details, see [22].

3.1.1 Arithmetic Fuchsian groups

Let A be a quaternion algebra over $\mathbb{Q}$ with Hilbert symbol $\left(\frac{a,b}{\mathbb{Q}}\right)$. That is, A is a 4-dimensional algebra over $\mathbb{Q}$ having basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ with multiplication defined so that 1 is a multiplicative identity, and

$$\mathbf{i}^2 = a \cdot 1\,,\ \mathbf{j}^2 = b \cdot 1\,,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$

where $a, b \in \mathbb{Q}^*$. A admits an anti-involution $x \longmapsto \overline{x}$ called *conjugation*. That is, $x \longmapsto \overline{x}$ is an involution of the vector space and $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$. Conjugation is given by

$$\overline{x} = \overline{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$$

The *(reduced) norm* and *(reduced) trace* of $x \in A$ are defined by $n(x) = x\overline{x}$ and $tr(x) = x + \overline{x}$ respectively. We will view tr and n both as maps to $\mathbb{Q}$.

We note that the quaternion algebra $A = \left(\frac{a,b}{\mathbb{Q}}\right)$ embeds into $M_2(\mathbb{Q}(\sqrt{a}))$ by

$$\rho(x) = \rho(x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) = \begin{pmatrix} x_0 + x_1\sqrt{a} & b(x_2 + x_3\sqrt{a}) \\ x_2 - x_3\sqrt{a} & x_0 - x_1\sqrt{a} \end{pmatrix}$$

With this embedding, we have that $tr(x) = Tr(\rho(x))$ and $n(x) = det(\rho(x))$ where Tr and det are the usual trace and determinant of a square matrix, respectively.

An *order* $\mathcal{O}$ in a quaternion algebra A over $\mathbb{Q}$ is a finitely generated $\mathbb{Z}$-module contained in A such that $\mathcal{O}$ spans A over $\mathbb{Q}$ ($\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = A$) and $\mathcal{O}$ is a ring with 1. Given an order $\mathcal{O}$, define $\mathcal{O}^1 = \{x \in \mathcal{O} \mid n(x) = 1\}$.

A useful result is the following [22].

Theorem 3.1.1. *If A is a quaternion algebra over $\mathbb{Q}$ and $\mathcal{O}$ is an order in A, then $P\rho\mathcal{O}^1$ is a finite co-area Fuchsian group. Moreover, $P\rho\mathcal{O}^1$ is co-compact if and only if A is a division algebra.*

From this theorem, we make the following definition. A Fuchsian group Γ is said to be *derived from a quaternion algebra* (defined over $\mathbb{Q}$) if Γ is conjugate into a subgroup of $P\rho\mathcal{O}^1$ of finite index, for some A and $\mathcal{O}$ as above. Any Fuchsian group commensurable in $PSL_2(\mathbb{C})$ with a group derived from a quaternion algebra is said to be *arithmetic*.

3.1.2 Bianchi groups and Fuchsian subgroups

Let d be a positive square-free integer, and let $k_d = \mathbb{Q}(\sqrt{-d})$. Let $\mathcal{O}_d$ be the ring of integers in k_d. That is,

$$\mathcal{O}_d = \{a + b\sqrt{-d}\,|\, a, b \in \mathbb{Z}\} \text{ if } d \equiv 1 \text{ or } 2 \pmod 4$$

and

$$\mathcal{O}_d = \{\frac{a + b\sqrt{-d}}{2}\,|\, a, b \in \mathbb{Z}\,,\ a \equiv b \pmod 2\ \} \text{ if } d \equiv 3 \pmod 4$$

A *Bianchi group* is any group of the form $PSL_2(\mathcal{O}_d)$ for some square-free $d \in \mathbb{Z}^+$. It is well known that the Bianchi groups all have finite co-volume and are non-co-compact (see [22]).

We will be interested in Fuchsian subgroups of the Bianchi groups. We now describe a certain integral invariant of (the commensurability class of) such subgroups that will be useful for us. Any circle $\mathcal{C}$ in $\widehat{\mathbb{C}}$ can be described by a triple (a, B, c) as the set of $z \in \mathbb{C}$ (and possibly the point ∞) such that

$$a|z|^2 + Bz + \overline{Bz} + c = 0$$

If $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C})$ is non-elementary, it can be shown (see [21]) that we may choose $a, c \in \mathbb{Z}$ and $B \in \mathcal{O}_d$. In this case, we will write $B = \frac{1}{2}(b_1 + b_2\sqrt{-d})$ where $b_1, b_2 \in \mathbb{Z}$, $b_1 \equiv b_2 \pmod 2$, and both congruent to 0 (mod 2) if $d \equiv 1, 2 \pmod 4$. We say that the triple (a, B, c) is *primitive* if

$$gcd(a, \frac{b_1}{2}, \frac{b_2}{2}, c) = 1 \text{ for } b_1 \equiv b_2 \equiv 0 \pmod 2$$

and

$$gcd(a, b_1, b_2, c) = 1 \text{ for } b_1 \equiv b_2 \not\equiv 0 \pmod 2$$

A primitive triple for a circle is unique up to sign. If (a, B, c) is a primitive triple for $\mathcal{C}$, we define the *discriminant* of $\mathcal{C}$ to be $\mathcal{D}(\mathcal{C}) = |B|^2 - ac \in \mathbb{Z}^+$. $\mathcal{D}(\mathcal{C}) > 0$ since, when $a \neq 0$, the radius of $\mathcal{C}$ is $\mathcal{D}(\mathcal{C})/a^2$.

We denote the set of circles represented by primitive triples in $\mathcal{O}_d$ by Σ_d. $PSL_2(\mathcal{O}_d)$ acts on Σ_d by $T \cdot \mathcal{C} = T(\mathcal{C})$. That is, the Mӧbius transformation represented by $T \in PSL_2(\mathcal{O}_d)$ takes the circle $\mathcal{C} \in \Sigma_d$ to the circle $T(\mathcal{C})$, and $T(\mathcal{C})$ is represented by a primitive triple.

Now define

$$\mathcal{H}_d = \left\{ \begin{pmatrix} a & B \\ \overline{B} & c \end{pmatrix} \mid a, c \in \mathbb{Z}\,,\; B \in \mathcal{O}_d\,,\; ac - |B|^2 < 0, \text{ and } (a, B, c) \text{ primitive} \right\}$$

Let $\Phi : \mathcal{H}_d \to \Sigma_d$ be the obvious 2 to 1 map. We see that $det(A) = -\mathcal{D}(\Phi(A))$. There is a natural action of $SL_2(\mathcal{O}_d)$ on $\mathcal{H}_d$ by $T \cdot A = TAT^*$, where $T^* = \overline{T^t}$. As $\{\pm I\}$ is contained in the kernel of the action, we can induce an action by $PSL_2(\mathcal{O}_d)$. This action clearly preserves determinants.

If $A \in \mathcal{H}_d$, a calculation shows that $\Phi(V \cdot A) = T \cdot \Phi(A)$, where $V = (T^{-1})^t$. Thus, Φ descends to a map from the orbit space of $\mathcal{H}_d$ to the orbit space of Σ_d. This in turn implies that the discriminant function on Σ_d is invariant under the action of $PSL_2(\mathcal{O}_d)$.

Now we note that for any $T \in PSL_2(\mathcal{O}_d)$, $Stab_{PSL_2(\mathcal{O}_d)}(T \cdot \mathcal{C}) = T(Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}))T^{-1}$. Since $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C})$ and $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}')$ are commensurable in $PSL_2(\mathcal{O}_d)$ if and only if they are conjugate in $PSL_2(\mathcal{O}_d)$, we obtain the following [33].

Theorem 3.1.2. *Suppose that* $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C})$ *and* $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}')$ *are non-elementary Fuchsian groups commensurable in* $PSL_2(\mathcal{O}_d)$. *Then* $\mathcal{D}(\mathcal{C}) = \mathcal{D}(\mathcal{C}')$.

Suppose $D \in \mathbb{Z}^+$ and that $\mathcal{C}_D \subset \widehat{\mathbb{C}}$ is a circle centered at the origin of $\mathbb{C}$ with radius $\sqrt{D}$. Note that $\mathcal{D}(\mathcal{C}_D) = D$. One can check (see [21]) that

$$Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_D) = \left\{ P \begin{pmatrix} \alpha & D\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathcal{O}_d \text{ and } |\alpha|^2 - D|\beta|^2 = 1 \right\}$$

The following theorem provides us with the arithmetic structure of the groups $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C})$ (see [21], [23]).

Theorem 3.1.3. *Suppose d is a square-free integer. Then, for every $D \in \mathbb{Z}^+$,* $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_D)$ *is a Fuchsian group derived from the quaternion algebra*

$$A = \left(\frac{-d, D}{\mathbb{Q}} \right)$$

Moreover, the groups $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_D)$ *and* $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_{D'})$ *are commensurable in* $PSL_2(\mathcal{O}_d)$ *if and only if* $D = D'$.

Proof. Let $\mathcal{O} \subset A$ be the order defined as follows. For $d \equiv 3 \pmod 4$ set

$$\mathcal{O} = \{x = \frac{x_0}{2} + \frac{x_1}{2}\mathbf{i} + \frac{x_2}{2}\mathbf{j} + \frac{x_3}{2}\mathbf{k} \mid x_0, x_1, x_2, x_3 \in \mathbb{Z} \text{ and}$$

$$x_0 \equiv x_1 \,,\; x_2 \equiv x_3 \pmod 2 \;\}.$$

and for $d \equiv 1$ or $2 \pmod 4$, set

$$\mathcal{O} = \{x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_0, x_1, x_2, x_3 \in \mathbb{Z}\}$$

When $d \equiv 1, 2 \pmod 4$, $\mathcal{O}$ is easily seen to be an order in A. When $d \equiv 3 \pmod 4$, $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-module and has $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. Further, one can check that $\rho(\mathcal{O}) = R$, where

$$R = \left\{ \begin{pmatrix} \alpha & D\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathcal{O}_d \right\}.$$

R is a ring with 1 since $\mathcal{O}_d$ is. Therefore, $\mathcal{O}$ is a ring with 1 and hence an order.

Now we see that $P\rho\mathcal{O}^1 = Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_D)$ and the first assertion follows. The second assertion is immediate from Theorem 3.1.2 and the fact that $\mathcal{D}(\mathcal{C}_D) = D$. □

From Theorems 3.1.1 and 3.1.3 one can obtain the following useful criteria for $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_D)$ to be co-compact (see [22]).

Theorem 3.1.4. *Suppose $d, D \in \mathbb{Z}^+$ with $d \geq 3$ prime, and D a quadratic non-residue (mod d). Then $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_D)$ is a co-compact Fuchsian group.*

Remark 3.1.1. Since the squaring endomorphism of $(\mathbb{Z}/d\mathbb{Z})^*$ is not surjective when $d \geq 3$, quadratic non-residues always exist.

3.1.3 Figure eight knot group

Since M_8 is hyperbolic, we have

$$M_8 \cong \mathbb{H}^3/\Gamma_8$$

where $\Gamma_8 \cong \pi_1(M_8)$ (as a standard abuse of notation, we are making no distinction between M_8 and its interior). By conjugating if necessary, we may assume that Γ_8 is an index 12 subgroup of $PSL_2(\mathcal{O}_3)$ and that

$$\Gamma_8 = \left\langle P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, P\begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \right\rangle$$

where $\omega^2 + \omega + 1 = 0$ (note that $\mathbb{Z}[\omega] = \mathcal{O}_3$) [35].

Theorem 3.1.5. *For each positive integer $D \equiv 2$ (mod 3), $Stab_{\Gamma_8}(\mathcal{C}_D)$ is a co-compact Fuchsian group. Moreover, $Stab_{\Gamma_8}(\mathcal{C}_D)$ and $Stab_{\Gamma_8}(\mathcal{C}_{D'})$ are commensurable in Γ_8 if and only if $D = D'$.*

Proof. Let D be as in the statement of the theorem. Clearly $Stab_{\Gamma_8}(\mathcal{C}_D) = Stab_{PSL_2(\mathcal{O}_3)}(\mathcal{C}_D) \cap \Gamma_8$, so that

$$|Stab_{PSL_2(\mathcal{O}_3)}(\mathcal{C}_D) : Stab_{\Gamma_8}(\mathcal{C}_D)| \leq |PSL_2(\mathcal{O}_3) : \Gamma_8| = 12$$

Therefore, $Stab_{\Gamma_8}(\mathcal{C}_D)$ is co-compact if and only if $Stab_{PSL_2(\mathcal{O}_3)}(\mathcal{C}_D)$ is. Since 2 is not a square (mod 3), Theorem 3.1.4 implies $Stab_{PSL_2(\mathcal{O}_3)}(\mathcal{C}_D)$ is co-compact. The second assertion follows from Theorem 3.1.3. □

We will need another fact concerning Γ_8 which is of an arithmetic nature. Given any integer $n \geq 2$, let R_n be the ring $\mathcal{O}_3/(n)$, where (n) is the principal ideal in $\mathcal{O}_3$ generated by n. Define a homomorphism

$$\Phi_n : PSL_2(\mathcal{O}_3) \to PSL_2(R_n)$$

which is reduction of the entries modulo (n). The kernel of this homomorphism is a finite index, normal subgroup of $PSL_2(\mathcal{O}_3)$ called the *principal congruence subgroup of level n*, denoted $\Gamma(n)$. Any subgroup of $PSL_2(\mathcal{O}_3)$ containing $\Gamma(n)$ for some $n \geq 2$ is called a *congruence subgroup*. The theorem concerning the figure eight knot group which we need is the following well known fact. We include a proof (this one due to Mark Baker) for completeness.

Theorem 3.1.6. Γ_8 *is a congruence subgroup of* $PSL_2(\mathcal{O}_3)$ *containing* $\Gamma(4)$.

Proof. Let $\Gamma_8' \subset PSL_2(\mathcal{O}_3)$ be the group

$$\Gamma_8' = \left\langle P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, P\begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, P\begin{pmatrix} 1 & 1+2\omega \\ 0 & 1 \end{pmatrix} \right\rangle$$

According to [35], $\Gamma_8 \subset \Gamma_8'$ is a subgroup of index 2. We prove the theorem by proving

i. For any subgroup $H \subset \Gamma(2)$ with $|\Gamma(2) : H| \leq 2$, we have $\Gamma(4) \subset H$,

ii. $\Gamma(2) \subset \Gamma_8'$.

This will suffice since by (*ii*), $|\Gamma(2) : \Gamma(2) \cap \Gamma_8| = |\Gamma(2) \cap \Gamma_8' : \Gamma(2) \cap \Gamma_8| \leq |\Gamma_8' : \Gamma_8| = 2$. So by (*i*), $\Gamma(2) \cap \Gamma_8$, and hence Γ_8, contains $\Gamma(4)$.

Proof of (i): We prove this by showing that $\Gamma(4) = \Gamma(2)^{(2)}$, where

$$\Gamma(2)^{(2)} =< \gamma^2 \,|\, \gamma \in \Gamma(2) > .$$

For if this holds, then for any subgroup $H \subset \Gamma(2)$ of index no more than 2, we have $\gamma^2 \in H$ for any $\gamma \in \Gamma(2)$. Therefore, $\Gamma(4) = \Gamma(2)^{(2)} \subset H$ as required.

We first note that $G = \Gamma(2)/\Gamma(2)^{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^n$, for some $n \in \mathbb{Z}^+ \cup \{0\}$, since $\Gamma(2)$ is finitely generated. A simple calculation shows that $\gamma^2 \in \Gamma(4)$,

$\forall \gamma \in \Gamma(2)$, so that $\Gamma(2)^{(2)} \subset \Gamma(4)$. Hence, $\Gamma(2)/\Gamma(4)$ is a quotient of G. From [30], we see that $|\Gamma(2) : \Gamma(4)| = 32$, so that $\Gamma(2)/\Gamma(4) \cong (\mathbb{Z}/2\mathbb{Z})^5$, and $n \geq 5$.

In [1], it is shown that $\Gamma(2) \cong \pi_1(S^3 \setminus L)$ where L is a five component link in S^3. Therefore, $(\Gamma(2))^{ab} \cong \mathbb{Z}^5$, where $(\Gamma(2))^{ab}$ denotes the abelianization of $\Gamma(2)$. Since G is abelian, the quotient map $\Gamma(2) \to G$ factors through $(\Gamma(2))^{ab}$. Therefore, $n \leq 5$ implying $n = 5$ and $\Gamma(2)^{(2)} = \Gamma(4)$. Thus, (i) follows.

Proof of (ii): We consider

$$K = core_{PSL_2(\mathcal{O}_3)}(\Gamma'_8) = \bigcap_{\gamma \in PSL_2(\mathcal{O}_3)} \gamma \Gamma'_8 \gamma^{-1} \lhd PSL_2(\mathcal{O}_3)$$

As $K \subset \Gamma'_8$, to prove *(ii)* it will suffice to show that $\Gamma(2) \subset K$. Since $\gamma \Gamma'_8 \gamma^{-1} = (\gamma g)\Gamma'_8(\gamma g)^{-1}$ for every $g \in \Gamma'_8$, we see that $K = \bigcap_{i=1}^{6} s_i \Gamma'_8 s_i^{-1}$ where $s_1, ..., s_6$ are coset representatives for Γ'_8 in $PSL_2(\mathcal{O}_3)$. We claim that it is possible to choose $s_1, ..., s_6$ to lie in $Stab_{PSL_2(\mathcal{O}_3)}(\infty)$, that is, we may assume that matrices for the s_i are upper triangular. To see this, we note that $|PSL_2(\mathcal{O}_3) : \Gamma'_8|$ is equal to the number of sheets in the orbifold cover $p : M_{\Gamma'_8} \to M_{PSL_2(\mathcal{O}_3)}$ which is the number of points in $p^{-1}(x)$ where x is a nonsingular point of $M_{PSL_2(\mathcal{O}_3)}$. If we choose the point x to lie in the cusp of $M_{PSL_2(\mathcal{O}_3)}$, then $p^{-1}(x)$ is contained in the cusp of $M_{\Gamma'_8}$ (note that each of these orbifolds has exactly one cusp since M_8 does). Now, the cusps of $M_{PSL_2(\mathcal{O}_3)}$ and $M_{\Gamma'_8}$ are $B/Stab_{PSL_2(\mathcal{O}_3)}(\infty)$ and $B/Stab_{\Gamma'_8}(\infty)$, respectively, where B denotes a sufficiently high horoball centered at ∞. Therefore,

$$|Stab_{PSL_2(\mathcal{O}_3)}(\infty) : Stab_{\Gamma'_8}(\infty)| = |p^{-1}(x)| = |PSL_2(\mathcal{O}_3) : \Gamma'_8|$$

and thus the claim holds.

We now note that the diagonal entries of an upper triangular matrix must be units and therefore equal to one of $1, \omega$, or $\overline{\omega}$. However, since $\omega \not\equiv 1 \pmod{(2)}$ we see that a matrix for any element of $Stab_{\Gamma(2)}(\infty)$ must have 1's on the diagonal. It follows then from the fact that $\mathcal{O}_3$ is generated over $\mathbb{Z}$ by 1 and ω that

$$Stab_{\Gamma(2)}(\infty) = \left\langle P\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, P\begin{pmatrix} 1 & 2\omega \\ 0 & 1 \end{pmatrix} \right\rangle$$

and therefore $Stab_{\Gamma(2)}(\infty) \subset \Gamma'_8$. Furthermore, we see that since $\Gamma(2)$ is normal in $PSL_2(\mathcal{O}_3)$ and because $s_i \in Stab_{PSL_2(\mathcal{O}_3)}(\infty)$, we have

$$s_i Stab_{\Gamma(2)}(\infty) s_i^{-1} \subset \Gamma(2) \cap Stab_{PSL_2(\mathcal{O}_3)}(\infty) = Stab_{\Gamma(2)}(\infty)$$

for each $i = 1, ..., 6$. Therefore, $Stab_{\Gamma(2)}(\infty) \subset K$.

Now we again use the fact that $\Gamma(2) \cong \pi_1(S^3 \setminus L)$. A Wirtinger presentation generates $\pi_1(S^3 \setminus L)$ by meridians [36], hence $\Gamma(2)$ is generated by parabolics. If follows that the generators of $\Gamma(2)$ are $PSL_2(\mathcal{O}_3)$ conjugates of elements in $Stab_{\Gamma(2)}(\infty)$. Since $K \lhd PSL_2(\mathcal{O}_3)$, these generators must lie in K. Thus, $\Gamma(2) \subset K$, and (*ii*) follows, completing the proof. □

3.1.4 Proof of Theorem 3.1.7

Assuming Theorem 3.2.1, we prove

Theorem 3.1.7. *There exists a compact, orientable, irreducible* 3*-manifold* M*, with torus boundary, having the following property. Given any positive integer,* n*, there exists a closed, orientable, immersed, essential surface* $F \looparrowright$

M with no incompressible annulus joining F and ∂M, such that F compresses in $M(\alpha)$ and $M(\beta)$ and $\Delta(\alpha, \beta) > n$.

Proof. Let $M = M_8$, which is a compact, orientable, irreducible 3-manifold with torus boundary. Let μ and λ denote the standard basis for $\pi_1(\partial M)$, so that slopes on ∂M are represented by co-prime pairs $(p, q) \in \mathbb{Z}^2$.

Now, let $p = 4$ and $q > n$ be any odd integer. By Theorem 3.2.1, there exists F, a closed, orientable, immersed totally geodesic surface in M which compresses in $M(p, q)$. Since $M(1, 0) = S^3$ is simply connected, we have that F compresses in $M(1, 0)$. We also have $\Delta((p, q), (1, 0)) = q > n$, so that to complete the proof, we need only prove that there exists no incompressible annulus in M with one boundary component in F and the other in ∂M. Such an annulus defines a free homotopy from an essential curve in F to an essential curve in ∂M. That is, an essential curve in F is peripheral. This is forbidden by Corollary 1.1.4, hence no such annulus exists, and the theorem follows. □

3.2 Compressions and normal closures

3.2.1 Compressions in $M_8(\alpha)$

In this section we prove the main theorem.

Theorem 3.2.1. *Suppose $4|p$ and $3 \nmid p$. Then, for any q with $gcd(p, q) = 1$ there exists infinitely many non-commensurable, closed, orientable, immersed, totally geodesic surfaces in M_8 which compress in $M_8(p, q)$.*

Remark 3.2.1. We say that two surfaces in M_8 are commensurable if their fundamental groups are commensurable in Γ_8.

Proof. Let μ and λ be elements in $Stab_{\Gamma_8}(\infty)$ representing the standard meridian-longitude basis for $\pi_1(\partial M_8)$. It can be shown (see e.g. [39]), that

$$\mu = P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \lambda = P\begin{pmatrix} 1 & 4\omega + 2 \\ 0 & 1 \end{pmatrix}$$

Let p and q be as in the statement of the theorem, and put

$$\sigma = \mu^p \lambda^q = P\begin{pmatrix} 1 & p + q(4\omega + 2) \\ 0 & 1 \end{pmatrix} = P\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$$

Van Kampen's Theorem implies

$$\pi_1(M_8(p, q)) \cong \Gamma_8 / \ll \sigma \gg$$

where $\ll \sigma \gg$ is the normal closure of $\{\sigma\}$ in Γ_8.

The strategy is now the following. We construct a sequence of distinct positive integers $D_k \equiv 2 \pmod 3$, and elements $g_k \in \ll \sigma \gg$ of the form

$$g_k = P\begin{pmatrix} \alpha_k & D_k \beta_k \\ \overline{\beta_k} & \overline{\alpha_k} \end{pmatrix} \neq Id$$

with $\alpha_k, \beta_k \in \mathcal{O}_3$. Writing $\Gamma_{D_k} = Stab_{\Gamma_8}(\mathcal{C}_{D_k})$, Theorem 3.1.5 implies $\{\Gamma_{D_k}\}$ is a sequence of non-commensurable (in Γ_8) co-compact Fuchsian subgroups of Γ_8. Therefore, as in Section 1.1.2, we obtain a sequence

$$\{f_k : S_{\Gamma_{D_k}} \looparrowright M_8\}$$

of pairwise non-commensurable, closed, orientable, immersed, totally geodesic surfaces in M_8 with $\psi \circ f_{k*}(\pi_1(S_{\Gamma_{D_k}})) = \Gamma_{D_k}$.

The description of $Stab_{PSL_2(\mathcal{O}_3)}(\mathcal{C}_D)$ given in Section 3.1.2 and the fact that

$$\Gamma_{D_k} = \Gamma_8 \cap Stab_{PSL_2(\mathcal{O}_3)}(\mathcal{C}_{D_k})$$

together imply that $g_k \in \Gamma_{D_k}$. So we see that g_k represents a non-trivial element of $\pi_1(S_{\Gamma_{D_k}})$ lying in $\ll \sigma \gg$, and hence, S_{D_k} compresses in $M_8(p,q)$.

We now begin the construction of D_k and g_k. We have

$$|\xi|^2 = (p + q(4\omega + 2))(p + q(4\overline{\omega} + 2)) = p^2 + 12q^2$$

Since $3 \nmid p$, we see that $3 \nmid |\xi|^2$, and $3 \nmid 4|\xi|^2$. Therefore, $gcd(3, 4|\xi|^2) = 1$ and there exists $r, t \in \mathbb{Z}$ such that

$$-3r - 4|\xi|^2 t = 1$$

This implies that

$$h = P\begin{pmatrix} \sqrt{-3} & 4\xi t \\ \bar{\xi} & \sqrt{-3}r \end{pmatrix}$$

is an element of $PSL_2(\mathcal{O}_3)$.

Claim: $h \in \Gamma_8$.

To prove the claim, we consider the homomorphism Φ_4 defining $\Gamma(4)$ (see Section 3.1.3).

By Theorem 3.1.6, we have that $\Gamma(4) \subset \Gamma_8$. In particular $\Gamma_8 = \Phi_4^{-1}(\Phi_4(\Gamma_8))$. So to prove that $h \in \Gamma_8$, it suffices to prove that $\Phi_4(h) \in \Phi_4(\Gamma_8)$. We have

$$h = P\begin{pmatrix} \sqrt{-3} & 4\xi t \\ \bar{\xi} & \sqrt{-3}r \end{pmatrix} \equiv P\begin{pmatrix} \sqrt{-3} & 0 \\ 2 & \sqrt{-3} \end{pmatrix} \pmod{4}$$

The congruence in the $(1,2)$-entry is clear. The congruence in the $(2,1)$-entry follows from the fact that $4|p$ and $gcd(p,q) = 1$ implies $q \equiv 1, 3 \pmod 4$, so that

$$\overline{\xi} = p + q(4\overline{\omega} + 2) \equiv 2q \equiv 2 \pmod{(4)}$$

The congruence in the $(2,2)$-entry comes from

$$1 = det(h) = -3r - 4|\xi|^2 t \equiv r \pmod{(4)}$$

Next, we consider the following element $g \in \Gamma_8$, and its reduction modulo (4)

$$g = P\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}\right)^2 = P\begin{pmatrix} 1 + 6\omega + 4\omega^2 & 4(1+\omega) \\ 2(-1) & 1 + 2\omega \end{pmatrix}$$

$$= P\begin{pmatrix} \sqrt{-3} - 4 & 2 + 2\sqrt{-3} \\ -2 & \sqrt{-3} \end{pmatrix} \equiv P\begin{pmatrix} \sqrt{-3} & 0 \\ 2 & \sqrt{-3} \end{pmatrix} \pmod{(4)}$$

Therefore, $\Phi_4(h) = \Phi_4(g)$, hence $\Phi_4(h) \in \Phi_4(\Gamma_8)$ and the claim is established.

We now construct D_k and g_k. For each $k \in \mathbb{Z}^+$ set

$$n_k = -3|\xi|^2(2 + 3k) + 9$$

$$D_k = |\xi|^2 n_k^2 + 2 + 3k$$

$$g_k = \sigma^{n_k}(h\sigma h^{-1})^6 \sigma^{n_k}$$

By construction, $g_k \in \ll \sigma \gg$ and $D_k \equiv 2 \pmod{3}$. To complete the proof of the theorem, we must show that g_k has the required form.

First, we compute

$$(h\sigma h^{-1})^6 = h\sigma^6 h^{-1} = P\left(\begin{pmatrix} \sqrt{-3} & 4\xi t \\ \overline{\xi} & \sqrt{-3}r \end{pmatrix}\begin{pmatrix} 1 & 6\xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{-3}r & -4\xi t \\ -\overline{\xi} & \sqrt{-3} \end{pmatrix}\right)$$

$$= P\left(\begin{pmatrix} \sqrt{-3} & 6\xi\sqrt{-3} + 4\xi t \\ \overline{\xi} & 6|\xi|^2 + \sqrt{-3}r \end{pmatrix}\begin{pmatrix} \sqrt{-3}r & -4\xi t \\ -\overline{\xi} & \sqrt{-3} \end{pmatrix}\right)$$

$$= P \begin{pmatrix} -3r - \overline{\xi}(6\xi\sqrt{-3} + 4\xi t) & -4\xi t\sqrt{-3} + \sqrt{-3}(6\xi\sqrt{-3} + 4\xi t) \\ \overline{\xi}\sqrt{-3}r - \overline{\xi}(6|\xi|^2 + \sqrt{-3}r) & -4|\xi|^2 t + \sqrt{-3}(6|\xi|^2 + \sqrt{-3}r) \end{pmatrix}$$

$$= P \begin{pmatrix} -3r - 4|\xi|^2 t - 6|\xi|^2\sqrt{-3} & -18\xi \\ -6|\xi|^2\overline{\xi} & -4|\xi|^2 t - 3r + 6|\xi|^2\sqrt{-3} \end{pmatrix}$$

$$= P \begin{pmatrix} 1 - 6|\xi|^2\sqrt{-3} & -18\xi \\ -6|\xi|^2\overline{\xi} & 1 + 6|\xi|^2\sqrt{-3} \end{pmatrix}$$

This gives

$$g_k = \sigma^{n_k}(h\sigma h^{-1})^6\sigma^{n_k}$$

$$= P \left(\begin{pmatrix} 1 & n_k\xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - 6|\xi|^2\sqrt{-3} & -18\xi \\ -6|\xi|^2\overline{\xi} & 1 + 6|\xi|^2\sqrt{-3} \end{pmatrix} \begin{pmatrix} 1 & n_k\xi \\ 0 & 1 \end{pmatrix} \right)$$

$$= P \left(\begin{pmatrix} 1 - 6|\xi|^2\sqrt{-3} - 6n_k|\xi|^4 & -18\xi + n_k\xi(1 + 6|\xi|^2\sqrt{-3}) \\ -6|\xi|^2\overline{\xi} & 1 + 6|\xi|^2\sqrt{-3} \end{pmatrix} \begin{pmatrix} 1 & n_k\xi \\ 0 & 1 \end{pmatrix} \right)$$

$$= P \begin{pmatrix} 1 - 6n_k|\xi|^4 - 6|\xi|^2\sqrt{-3} & n_k\xi(2 - 6n_k|\xi|^4) - 18\xi \\ -6|\xi|^2\overline{\xi} & 1 - 6n_k|\xi|^4 + 6|\xi|^2\sqrt{-3} \end{pmatrix}$$

Now set

$$\alpha_k = 1 - 6n_k|\xi|^4 - 6|\xi|^2\sqrt{-3}$$

$$\beta_k = -6|\xi|^2\xi$$

and note that

$$n_k\xi(2 - 6n_k|\xi|^4) - 18\xi = 2n_k\xi - 6n_k^2|\xi|^4\xi - 18\xi$$

$$= (-3|\xi|^2(2 + 3k) + 9)2\xi - |\xi|^2 n_k^2 \cdot 6|\xi|^2\xi - 18\xi$$

$$= (2 + 3k)(-6|\xi|^2\xi) + 18\xi + |\xi|^2 n_k^2(-6|\xi|^2\xi) - 18\xi$$

$$= (|\xi|^2 n_k^2 + 2 + 3k)(-6|\xi|^2\xi) = D_k\beta_k$$

Thus we have

$$g_k = P\begin{pmatrix} 1-6n_k|\xi|^4-6|\xi|^2\sqrt{-3} & n_k\xi(2-6n_k|\xi|^4)-18\xi \\ -6|\xi|^2\overline{\xi} & 1-6n_k|\xi|^4+6|\xi|^2\sqrt{-3} \end{pmatrix}$$
$$= P\begin{pmatrix} \alpha_k & D_k\beta_k \\ \overline{\beta_k} & \overline{\alpha_k} \end{pmatrix}$$

This completes the proof. □

The element h in the above proof was arrived at by attempting to "match up" an invariant hyperbolic plane of σ with one from a conjugate of σ. The group generated by σ and $h\sigma h^{-1}$ is a non-elementary Fuchsian subgroup of Γ_8, which thus contains a rank-2 free subgroup of hyperbolic elements in $\ll \sigma \gg$ with real traces. The fact that Γ_8 contains an abundance of co-compact Fuchsian subgroups makes it possible to construct nontrivial elements in the intersection of some of these groups with $\ll \sigma \gg$.

It seems worthwhile to compute h and g_k explicitly for some choice of p, q, and k. Section 3.3 at the end of this chapter contains h and a list of g_k and D_k for $k \in \{1, ..., 10\}$, when $p = 20$ and $q = 7$.

3.2.2 Related results

The basic construction in the proof of Theorem 3.2.1 is quite general. In particular, we have

Theorem 3.2.2. *Let $d \geq 3$ with d prime. Suppose also that*

$$\sigma = P\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \in PSL_2(\mathcal{O}_d)$$

is such that $d \nmid |\xi|^2$. Then $\ll \sigma \gg$ non-trivially intersects infinitely many non-commensurable (in $PSL_2(\mathcal{O}_d)$) co-compact Fuchsian subgroups of $PSL_2(\mathcal{O}_d)$.

We only sketch the proof as it is almost identical to the proof of Theorem 3.2.1.

Sketch of Proof. As in the previous proof, we construct, for each $k \in \mathbb{Z}^+$, $g_k \in \ll \sigma \gg$ of the form

$$g_k = P \begin{pmatrix} \alpha_k & D_k \beta_k \\ \overline{\beta_k} & \overline{\alpha_k} \end{pmatrix}$$

where $\{D_k\}$ is a sequence of distinct positive integers, each of which is a quadratic non-residue (mod d). Therefore, $Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_{D_k})$ is a co-compact Fuchsian group by Theorem 3.1.4, and $g_k \in \ll \sigma \gg \cap Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_{D_k})$. Moreover, the groups in the sequence $\{Stab_{PSL_2(\mathcal{O}_d)}(\mathcal{C}_{D_k})\}$ are pairwise non-commensurable in $PSL_2(\mathcal{O}_d)$ by Theorem 3.1.3, as required.

By hypothesis, there exists r and t such that $-dr - |\xi|^2 t = 1$. This implies that

$$h = P \begin{pmatrix} \sqrt{-d} & \xi t \\ \overline{\xi} & \sqrt{-d} r \end{pmatrix} \in PSL_2(\mathcal{O}_d)$$

Let $x < d$ be a non-square (mod d) (see the remark at the end of Section 3.1.2). For each $k \in \mathbb{Z}^+$ define

$$n_k = -d|\xi|^2(dk + x) + d^2$$

$$D_k = n_k^2 |\xi|^2 + dk + x$$

$$g_k = \sigma^{n_k} (h \sigma h^{-1})^{2d} \sigma^{n_k}$$

$$\alpha_k = 1 - 2dn_k|\xi|^4 - 2d|\xi|^2\sqrt{-d}$$

$$\beta_k = -2d|\xi|^2 \xi$$

By definition, $D_k \equiv x$ (mod d), and hence is not a square (mod d). As in the previous proof, a calculation shows

$$g_k = P \begin{pmatrix} \alpha_k & D_k\beta_k \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

thus completing the proof. □

Theorem 3.2.2 does not quite have the same topological implications as Theorem 3.2.1, since $M_{PSL_2(\mathcal{O}_d)}$ is never a manifold (i.e. $PSL_2(\mathcal{O}_d)$ always contains torsion).

We wish to use Theorem 3.2.2 to construct other examples of cusped hyperbolic manifolds and arbitrarily large surgeries in which totally geodesic surfaces compress. Selberg's Lemma (see [31], for example) guarantees the existence of a hyperbolic 3-manifold M_Γ that is a finite sheeted orbifold cover of $M_{PSL_2(\mathcal{O}_d)}$ (pass to $\Gamma \subset PSL_2(\mathcal{O}_d)$ a finite index, torsion free subgroup). The problem in trying to use these covers is that the conjugating element h in the proof of Theorem 3.2.2 need not lie in Γ.

There is a special situation of torsion free, finite index subgroups of $PSL_2(\mathcal{O}_d)$ where Theorem 3.2.2 can be applied.

Suppose $d > 3$ is prime. In this situation, note that the peripheral subgroup $Stab_{PSL_2(\mathcal{O}_d)}(\infty)$ is a rank-2 torsion free abelian group (the only units of $\mathcal{O}_d$ are ± 1 for $d > 3$). We say that a finite index subgroup $\Gamma \subset PSL_2(\mathcal{O}_d)$ is ∞-*non-separated* if we can choose coset representatives for $\Gamma \subset PSL_2(\mathcal{O}_d)$ from $Stab_{PSL_2(\mathcal{O}_d)}(\infty)$. In this situation, we will denote such a set of coset representatives by $s_1, ..., s_n$. This condition is equivalent to requiring that the index of $Stab_\Gamma(\infty)$ in $Stab_{PSL_2(\mathcal{O}_d)}(\infty)$ is equal to the index of Γ in $PSL_2(\mathcal{O}_d)$. This is also equivalent to the assumption that the preimage of the cusp of $M_{PSL_2(\mathcal{O}_d)}$ corresponding to $Stab_{PSL_2(\mathcal{O}_d)}(\infty)$ is a single cusp of M_Γ.

Suppose now that $d > 3$ and $\Gamma \subset PSL_2(\mathcal{O}_d)$ is ∞-non-separated. Note that any element $h \in PSL_2(\mathcal{O}_d)$ can be written as gs_i for some $g \in \Gamma$ and some $i \in \{1, ..., n\}$. Let σ be an arbitrary element of $Stab_\Gamma(\infty)$. Since each s_i centralizes σ, we see that

$$h\sigma h^{-1} = gs_i\sigma s_i^{-1}g^{-1} = g\sigma g^{-1}$$

It follows that $\ll \sigma \gg_\Gamma = \ll \sigma \gg_{PSL_2(\mathcal{O}_d)}$ (here, $\ll \sigma \gg_G$ denotes the normal closure of σ in G).

Example 3.2.1. It can be shown that there exist torsion free subgroups Γ_0 and Γ_1 in $PSL_2(\mathcal{O}_7)$, each of index 6 (see [13] where these groups are called $\Gamma_{-7}(6, 8)$ and $\Gamma_{-7}(6, 9)$). We show that for each $i = 0, 1$, there are infinitely many surgeries on M_{Γ_i}, such that for each such surgered manifold, $M_{\Gamma_i}(\alpha)$, there are infinitely many non-commensurable closed, orientable, immersed, totally geodesic surfaces in M_{Γ_i} which compress in $M_{\Gamma_i}(\alpha)$.

Throughout, let $i = 0, 1$. In [13] it is shown that $H_1(M_{\Gamma_i}; \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2$. A well known homological argument shows that a compact 3-manifold M with boundary must satisfy

$$b_1(M) \geq \frac{b_1(\partial M)}{2}.$$

It follows that M_{Γ_i} can have no more than one cusp. Since, M_{Γ_i} must have at least one cusp (namely the one coming from $Stab_{\Gamma_i}(\infty)$), we see that M_{Γ_i} has exactly one cusp, which implies $\Gamma_i \subset PSL_2(\mathcal{O}_7)$ is ∞-non-separated.

We take the basis for $Stab_{PSL_2(\mathcal{O}_d)}(\infty)$ given by

$$\mu = P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \lambda = P\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

where $\eta = \frac{1+\sqrt{-7}}{2}$. Then any primitive element

$$\sigma = \mu^p \lambda^q = P\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \in Stab_{PSL_2(\mathcal{O}_d)}(\infty)$$

has $\xi = p + q\eta$ with (p, q) a pair of co-prime integers. This implies

$$|\xi|^2 = p^2 + pq + 2q^2$$

We assume that either $7|p$ or $7|q$, so that $7 \nmid |\xi|^2$.

Taking σ^{n_i} to be the smallest power of σ which lifts to $Stab_{\Gamma_i}(\infty)$ we see that n_i divides $6 = |Stab_{PSL_2(\mathcal{O}_d)}(\infty) : Stab_{\Gamma}(\infty)|$. We can write

$$\sigma^{n_i} = \begin{pmatrix} 1 & n_i\xi \\ 0 & 1 \end{pmatrix} \text{ for } i = 0, 1$$

Since $7 \nmid |\xi|^2$, we see that $7 \nmid |n_i\xi|^2$, so that σ^{n_i} satisfies the hypothesis of Theorem 3.2.2 and hence $\ll \sigma^{n_i} \gg_{\Gamma_i} = \ll \sigma^{n_i} \gg_{PSL_2(\mathcal{O}_7)}$ intersects infinitely many non-commensurable co-compact Fuchsian subgroups. As in the proof of Theorem 3.2.1, we see that this implies that there exist infinitely many non-commensurable closed, orientable, immersed, totally geodesic surfaces in M_{Γ_i} which compress in $M_{\Gamma_i}(\sigma^{n_i})$.

This example also proves Theorem 3.1.7. To see this, we refer to [13] for a presentation of Γ_0 $(= \Gamma_{-7}(6, 8))$:

$$\Gamma_0 = < g_1, g_2, g_3 \,|\, g_1 g_2^{-1} g_3 g_2^{-1} g_3^{-1} g_1^{-1} g_2^{-1},\ g_1 g_2 g_3^{-1} g_1 g_3^{-1} g_2^{-1} g_1 g_3^{-1} >$$

It is also shown there that $g_2 = \mu^2$. From this presentation, we see that g_2 is primitive in the abelianization, hence it is primitive in Γ_0. Furthermore, a calculation shows $\Gamma_0/ \ll g_2 \gg \cong \mathbb{Z}*\mathbb{Z}/3\mathbb{Z}$. As this group can never contain the fundamental group of a closed surface (of positive genus), we see that every one of the surfaces found above compresses in $M_{\Gamma_0}(g_2)$. Therefore, if in the above construction we let $p = 1$ and $q = 7k$ for $k \in \mathbb{Z}^+$, the sequence of slopes $(\sigma_k)^{n_{0,k}} = (\mu\lambda^{7k})^{n_{0,k}}$ have totally geodesic surfaces compressing in $M_{\Gamma_0}(\sigma_k^{n_{0,k}})$, and

$$7k = \Delta(\sigma_k, \mu) \leq \Delta(\sigma_k^{n_{0,k}}, g_2) \to \infty \text{ as } k \to \infty.$$

3.3 Examples

Here is a list of g_k and D_k from the proof of Theorem 3.2.1, for $p = 20$ and $q = 7$ where $k \in \{1, ..., 10\}$.

First, we have

$$h = \begin{pmatrix} \sqrt{-3} & -80 - 56\sqrt{-3} \\ 20 - 14\sqrt{-3} & 1317\sqrt{-3} \end{pmatrix}$$

k	D_k	g_k
1	216733332353	$\begin{pmatrix} 86746012705 - 5928\sqrt{-3} & -25695903883771680 - 17987132718640176\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 86746012705 + 5928\sqrt{-3} \end{pmatrix}$
2	555090222500	$\begin{pmatrix} 138825247393 - 5928\sqrt{-3} & -65811496779600000 - 46068047745720000\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 138825247393 + 5928\sqrt{-3} \end{pmatrix}$
3	1049684816711	$\begin{pmatrix} 190904482081 - 5928\sqrt{-3} & -124450631869256160 - 87115442308479312\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 190904482081 + 5928\sqrt{-3} \end{pmatrix}$
4	1700517114986	$\begin{pmatrix} 242983716769 - 5928\sqrt{-3} & -201613309152740160 - 141129316406918112\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 242983716769 + 5928\sqrt{-3} \end{pmatrix}$
5	2507587117325	$\begin{pmatrix} 295062951457 - 5928\sqrt{-3} & -297299528630052000 - 208109670041036400\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 295062951457 + 5928\sqrt{-3} \end{pmatrix}$

6 3470894823728 $\begin{pmatrix} 347142186145 - 5928\sqrt{-3} & -411509290301191680 - 288056503210834176\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 347142186145 + 5928\sqrt{-3} \end{pmatrix}$

7 4590440234195 $\begin{pmatrix} 399221420833 - 5928\sqrt{-3} & -544242594166159200 - 380969815916311440\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 399221420833 + 5928\sqrt{-3} \end{pmatrix}$

8 5866223348726 $\begin{pmatrix} 451300655521 - 5928\sqrt{-3} & -695499440224954560 - 486849608157468192\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 451300655521 + 5928\sqrt{-3} \end{pmatrix}$

9 7298244167321 $\begin{pmatrix} 503379890209 - 5928\sqrt{-3} & -865279828477577760 - 605695879934304432\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 503379890209 + 5928\sqrt{-3} \end{pmatrix}$

10 8886502689980 $\begin{pmatrix} 555459124897 - 5928\sqrt{-3} & -1053583758924028800 - 737508631246820160\sqrt{-3} \\ -118560 + 82992\sqrt{-3} & 555459124897 + 5928\sqrt{-3} \end{pmatrix}$

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