One-ended subgroups of right–angled Artin groups

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1 Introduction

The motivating question for this short note is the following [Be, Question 12.3]. Let Mod(S) denote the mapping class group of a hyperbolic surface S of finite type (not homeomorphic to the once-punctured torus, or three- or four-times-punctured sphere).

Question 1.1 (M. Kapovich)**.** *Are there* 1*–ended subgroups* $G < \mathrm{Mod}(S)$ *which are purely pseudo-Anosov? Are there purely pseudo-Anosov surface subgroups* $G < \mathrm{Mod}(S)$*?*

A surface group is a group isomorphic to the fundamental group of a closed surface of genus at least 2.

Right–angled Artin groups frequently contain many closed surface subgroups as well as other interesting one-ended subgroups including hyperbolic 3– and 4–manifold groups; see [SDS, CW1, CW2, CF]. With the abundance of injections of right–angled Artin groups into mapping class groups—see [CP], [CW1], [CW2] and [CF]—one is tempted to start searching for an example to answer Question 1.1 within this class of groups. Our main theorem says this will not yield any such example.

Theorem 1.2. *Suppose* $G < A < \mathrm{Mod}(S)$ *where* S *is any finite type surface,* A *is isomorphic to a right–angled Artin group and* G *is a finitely presented* 1*–ended subgroup. Then* G *is not purely pseudo-Anosov.*

It should be mentioned that the constructions of embeddings of right–angled Artin groups referred to above can be carried out so as to contain pseudo-Anosov mapping classes (indeed, free purely pseudo-Anosov subgroups) as long as the complementary graph of the defining graph for the Artin group is connected—for example, the right–angled Artin group associated to the pentagon embeds with image containing pseudo-Anosov mapping classes.

Theorem 1.2 follows immediately from the next purely algebraic fact about 1–ended subgroups of right–angled Artin groups.

Theorem 1.3. *If* A *is a right–angled Artin group and* $G < A$ *a finitely presented* 1*–ended subgroup, then there exists* $g \in G$ *contained in a rank two free abelian subgroup of* A*.*

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Proof of Theorem 1.2 assuming Theorem 1.3. Let $g \in G$ be as in Theorem 1.3. Then the centralizer of g in $\mathrm{Mod}(S)$ contains $\mathbb{Z}^2$, and so g cannot be pseudo-Anosov; see [Iv, §8]. $\square$

The proof of Theorem 1.3 that we give below requires only elementary geometric topology. The case of G being a surface group can also be assembled from various facts proved in [CW1].

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2 Proof

Let $s_1, ..., s_n$ be the standard generators for A. Let T_A denote the locally CAT(0) cubed complex with $\pi_1(T_A) = A$ described, for example, in [CW1, p.446]. This is obtained from the standard n–torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by appropriately removing interiors of k–sub-cube faces whenever the corresponding k generators do not all commute (see also [Ch] where this is called the *Salvetti complex*). By construction, we have an inclusion $T_A \subseteq \mathbb{T}^n$.

For each generator, s_i, there is a "dual hypersurface" T_i constructed as follows. Consider the affine codimension one hyperplane

$$\mathbb{A}_i = \mathbb{R}^{i-1} \times \{1/2\} \times \mathbb{R}^{n-i} \subset \mathbb{R}^n$$

This is stabilized by $\mathbb{Z}^{i-1} \times \{0\} \times \mathbb{Z}^{n-i} < \mathbb{Z}^n$. The quotient embeds as nonseparating codimension–one torus

$$\mathbb{T}_i \subset \mathbb{T}^n.$$

We define

$$T_i = T_A \cap \mathbb{T}_i.$$

T_i is bicollared in T_A, with a compact neighborhood $N(T_i) \cong T_i \times [-1, 1]$. An important point about T_i is that the elements of $\pi_1(T_i) < \pi_1(T_A)$ all commute with s_i. Indeed, any element $g \in \pi_1(T_i)$ and s_i generate a rank two free abelian subgroup.

Proof of Theorem 1.3. Let X be the presentation 2–complex with $\pi_1(X) = G$. Let

$$f : X \to T_A$$

be a map for which $f_* : G = \pi_1(X) \to \pi_1(T_A) = A$ is given by inclusion $G < A$, up to conjugation.

We apply a homotopy so that $f \pitchfork T_1$ and we assume that f has been homotoped so that $|f^{-1}(T_1)|$, the number of components of $f^{-1}(T_1)$, is minimized. If $f^{-1}(T_1)$ is empty, then $G < A'$, where A' is the right–angled Artin group on the generators $s_2, ..., s_n$ (with the same relations), and we can apply induction on the number of generators of A. We suppose therefore that $f^{-1}(T_1)$ is nonempty.

Now $f^{-1}(T_1)$ is a graph in X which locally separates X. Indeed, by transversallity, $f^{-1}(T_1)$ has a compact neighborhood homeomorphic to a product $N(f^{-1}(T_1)) \cong f^{-1}(T_1) \times [-1,1]$. Let Y denote any component of $f^{-1}(T_1)$. There are a two cases to consider.

Case 1. Y contains a homotopically nontrivial loop γ.

In this case, $f(\gamma) \subset T_1$, and so $g = f_*([\gamma]) \in \pi_1(T_1) < \pi_1(T) = A$ commutes with the generator s_1. Then g and s_1 generate a rank two free abelian subgroup of A, as required.

Case 2. Every loop in Y is homotopically trivial.

In this case, we can lift Y to the universal covering

$$\widetilde{Y} \subset \widetilde{X}.$$

Now $\widetilde{Y}$ is a connected compact graph which locally separates $\widetilde{X}$. We claim that $\widetilde{Y}$ must in fact globally separate $\widetilde{X}$. Indeed, the bicollaring of Y lifts to one on $\widetilde{Y}$, and hence $\widetilde{Y}$ has a neighborhood homeomorphic to a product $N(\widetilde{Y}) \cong \widetilde{Y} \times [-1,1]$. If $\widetilde{Y}$ does not globally separate, then cutting open along $\widetilde{Y}$, and gluing infinitely many copies of the resulting space end-to-end constructs an infinite cyclic cover of $\widetilde{X}$ (here we are using the bicollaring), which is impossible since $\widetilde{X}$ is simply connected.

So, $\widetilde{Y}$ separates $\widetilde{X}$ into two components. If both components are unbounded, then $\widetilde{X}$ has at least 2 ends which contradicts the fact that G has one end. Therefore, we can assume that one of the components call it $\widetilde{Z} \subset \widetilde{X} \setminus \widetilde{Y}$ is bounded.

We observe that $N(\widetilde{Y}) \cup \widetilde{Z}$ is compact since $N(\widetilde{Y})$ is compact and $\widetilde{Z}$ is bounded with closure $\widetilde{Y} \cup \widetilde{Z}$.

Claim 2.1. *The covering projection $p : \widetilde{X} \to X$ restricted to $N(\widetilde{Y}) \cup \widetilde{Z}$ is injective.*

Proof of claim. Note that $p|_W$ being injective (for any set W) is equivalent to the property that any nontrivial covering transformation takes W disjoint from itself. Therefore, any nontrivial covering transformation takes $N(\widetilde{Y})$ disjoint from itself. So, if a nontrivial covering transformation γ has $\gamma\widetilde{Z} \cap \widetilde{Z} \neq \emptyset$, then after replacing γ with γ^{-1} if necessary, we may assume that $\gamma\widetilde{Z} \subset \widetilde{Z}$. It follows that $\gamma^k(\widetilde{Z}) \subset \widetilde{Z}$ for all $k \geq 1$. Since $G < A$ is torsion free and acts properly discontinuously on $\widetilde{X}$, the compactness of $\widetilde{Y} \cup \widetilde{Z}$ makes this impossible, providing a contradiction. □

We let $Z = p(\widetilde{Z})$ so that $p : N(\widetilde{Y}) \cup \widetilde{Z} \to N(Y) \cup Z$ is a homeomorphism by the claim. We can homotope $f : X \to T$ to a map $h : X \to T$ with a homotopy supported on $N(Y) \cup Z$ so that $h(N(Y) \cup Z)$ which pushes $N(Y) \cup Z$ off of T_1. Because the homotopy does not change the map outside of $N(Y) \cup Z$, we have $|h^{-1}(T_1)| < |f^{-1}(T_1)|$. This contradicts our assumption that the number of components of $f^{-1}(T_1)$ was minimal among maps inducing the inclusion $G < A$ on fundamental groups. Therefore, Case 2 cannot occur, and we are done. □

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