Surgeries on one component of the Whitehead link are virtually fibered

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Abstract

In this paper we will show that every Dehn filling on one component of the boundary of the exterior of the Whitehead link is virtually fibered. As a corollary we produce what seem to be the first examples of knot exteriors in S^3 which are virtually fibered but not fibered.

1 Introduction

Let M be a compact orientable 3-manifold with boundary consisting of a (possibly empty) union of tori. M is said to be *fibered* if it can be given the structure of a surface bundle over the circle. If M is finitely covered by a fibered manifold, then M is said to be *virtually fibered*. M is said to be *hyperbolic* if the interior of M admits a complete hyperbolic structure. Let L be a link in S^3. Denote by $N(L)$ the interior of a regular neighborhood of L in S^3, and $X(L) = S^3 \setminus N(L)$ the *exterior* of L. Manifolds (or more generally orbifolds) M and N are said to be *commensurable* if they share a finite sheeted (orbifold) cover. It is clear that commensurability is an equivalence relation. Since fibering persists in finite covers, the property of being virtually fibered is an invariant of the commensurability class.

If M is a Seifert fibered manifold, it is known (see [5] for example) that M is virtually fibered if and only if either the rational Euler number of the Seifert fibration or the orbifold Euler characteristic of the base is zero. In [17] Thurston asked if every (finite volume) hyperbolic manifold is virtually fibered.

Thurston's question is conjectured to have an affirmative answer. In general, however, finding examples of hyperbolic 3-manifolds which are not fibered, but are virtually fibered, is a hard problem. An elementary construction is to take M to be the union of two twisted I-bundles over a surface Σ glued together along their ∂I-bundles. There is an obvious 2-fold cover which is fibered (each of the I-bundles is covered by a product I-bundle and these can be glued together to obtain a bundle cover of M). For $\chi(\Sigma) < 0$ the gluing map can be chosen so that M is hyperbolic and so that M is not fibered (see [13] or [5]). In [5] Gabai gives examples of nonfibered virtually fibered link complements (with 2 or more components), and shows that they are not obtained by the above construction. In the same paper, Gabai also describes how to construct closed Haken manifolds with the same properties. In [13], Reid gives examples of non-Haken (in particular, nonfibered) virtually fibered closed hyperbolic 3-manifolds. Because the manifolds are non-Haken, they cannot be the union of two twisted I-bundles.

Let W be the Whitehead link, $\partial_0 X(W)$ one component of $\partial X(W)$, and $X(W)(p,q)$ the result of (p,q) Dehn filling on $\partial_0 X(W)$ (with respect to the meridian and longitude shown in Fig. 2.1). Our main result is

Theorem 2.3 *$X(W)(p,q)$ is virtually fibered.*

Combining this with the following result of Hodgson, Meyerhoff, and Weeks [6], we see that quite often the Dehn fillings produce manifolds which are not fibered but are virtually fibered.

Theorem 1.1 *$X(W)(p,q)$ is fibered if and only if $q = \pm 1$.*

In particular, this provides what seems to be the first known examples of nonfibered virtually fibered hyperbolic knot exteriors in S^3.

Let K_k denote the k-twist knot. K_5 is shown in Fig. 1.1.

Corollary 5.1 *$X(K_k)$ is virtually fibered for all $k \in \mathbb{Z}^+$. Moreover, when $k > 2$, $X(K_k)$ is hyperbolic and not fibered.*

Remarks: 1. The twist knots that we are considering are actually the reflections of those in the knot tables [14].

2. Since $H_1(X(K_k), \mathbb{Z}) \cong \mathbb{Z}$, it is easy to see that these are not the

union of two twisted I-bundles glued along their ∂I-bundles.

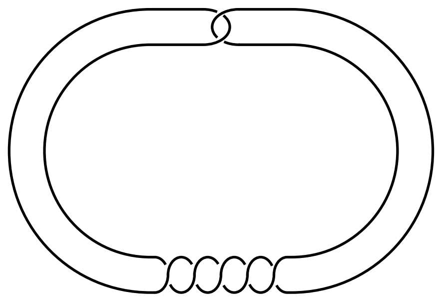

Figure 1.1

We will prove Theorem 2.3 by proving the following stronger result (see section 2 for the proof that Theorem 4.1 implies Theorem 2.3).

Let $C(p,s)$ denote the p-component chain link with s left handed half-twists (when s is negative, we will interpret this as meaning $-s$ right handed half twists). $C(6,-4)$ is shown in Fig. 2.2.

Theorem 4.1 *$X(C(p,s))$ is virtually fibered if and only if $(p,s) \neq (2,-1)$.*

The paper is organized as follows: Section 2 contains a proof of a result of Neumann and Reid [11] which determines a commensurability relationship between pairs of chain link exteriors needed to prove Theorem 4.1. In section 3 we recall the notion of the Murasugi sum of oriented surfaces in S^3 and a theorem of Gabai [4] which allows us to detect fibered link complements. In section 4 we prove Theorem 4.1. The idea is that half of the chain links are fibered, and the other half are commensurable with the first half, hence are virtually fibered. In the last section we briefly discuss applications of Theorem 2.3.

I would like to thank the referee for his comments and my advisor Alan Reid for the many helpful conversations and suggestions.

2 Commensurability of Certain Chain Links

Let W be the Whitehead link whose exterior is shown in Fig. 2.1, and $\partial_0 X(W) \subset \partial X(W)$ be the component containing m and l shown in

Fig. 2.1. We will take m and l as our prefered basis for $H_1(\partial_0 X(W), \mathbb{Z})$. Given a projection of $C(p,s)$ with the components arranged in circle (e.g. as in Fig. 2.2) we will refer to a *clasp* as a pair of crossings where the two ends of "adjacent" components (or of the same component in the case $p = 1$) are linked.

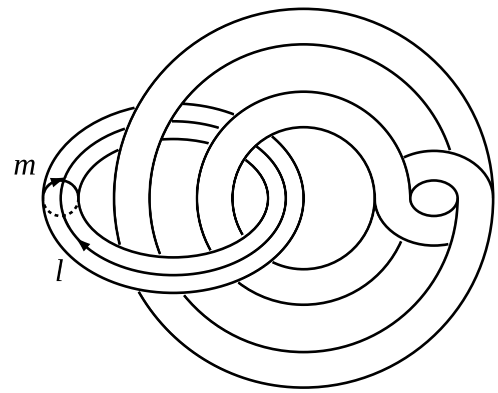

Figure 2.1

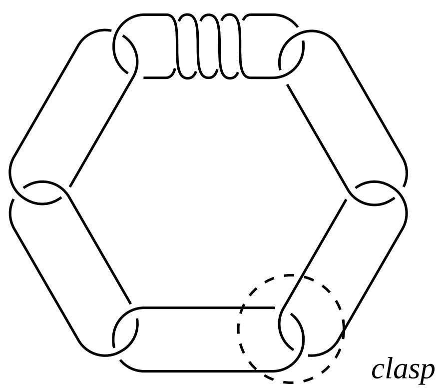

Figure 2.2

We recall some basic terminology about Dehn filling and orbifolds (see [11] and [17]). Given an orbifold M with singular locus Σ, let $\partial_0 M \subset \partial M$ be such that $\partial_0 M \cap \Sigma = \emptyset$ and $\partial_0 M \cong T^2$. Let m and l be generators for $H_1(\partial_0 M, \mathbb{Z})$. If p and q are relatively prime integers, then (p,q) Dehn filling on $\partial_0 M$ is the orbifold obtained by gluing a solid torus $S^1 \times D^2$ to $\partial_0 M$ along $\partial(S^1 \times D^2)$ so that $* \times \partial D^2$ represents $pm + ql$ in $H_1(\partial_0 M, \mathbb{Z})$. If p and q are not relatively prime, with $gcd(p,q) = d$, then (p,q) Dehn filling on $\partial_0 M$ will be the orbifold obtained by first $(p/d, q/d)$ Dehn filling on $\partial_0 M$ then giving the core curve a transverse angle of $2\pi/d$, so that it becomes part of the singular locus of the new orbifold with local group $\mathbb{Z}/d\mathbb{Z}$.

A relevant example of an orbifold is the *pillowcase* which is obtained from a torus T by an involution τ that we can view as an order 2 rotation about the line through T as indicated in Fig 2.3. We remark that $\tau_* = -(Identity)$ on $H_1(T, \mathbb{Z})$. The pillowcase has a 2-sphere as its underlying space, with the singular locus consisting of 4 points, each having local group $\mathbb{Z}/2\mathbb{Z}$.

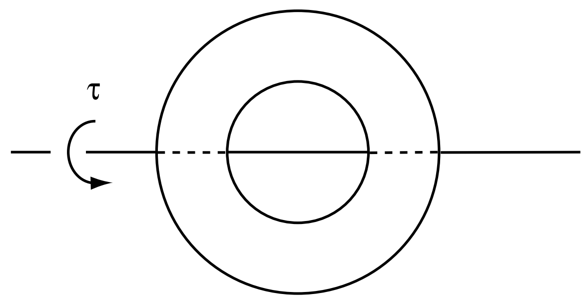

Figure 2.3

One way of proving that two orbifolds are commensurable is the following.

Lemma 2.1 *If M and N are finite sheeted orbifold covers of an orbifold P, then M and N are commensurable.*

Proof. By hypothesis, $\pi_1^{orb}(M)$ and $\pi_1^{orb}(N)$ inject into $\pi_1^{orb}(P)$, both with finite index. The intersection of their images is a finite index subgroup of $\pi_1^{orb}(P)$. The corresponding orbifold is a finite sheeted covering of both M and N.$\square$

The following theorem of Neumann and Reid [11] will be important in our work. We include its proof for completeness.

Theorem 2.2 *Let $p \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$. If $p + 2s \neq 0$, then $X(C(p, s))$ and $X(C(\pm(p + 2s), \mp(p + s)))$ are commensurable. Moreover, if s is even and $gcd(p, \frac{s}{2}) = 1$, then $X(C(p, s))$ is a p-fold cover of $X(W)(p, \frac{s}{2})$.*

Remark: The sign in $X(C(\pm(p + 2s), \mp(p + s)))$ is whichever makes the first entry positive.

Proof. We will prove that $X(C(p,s))$ and $X(C(\pm(p+2s),\mp(p+s)))$ cover of the same orbifold, hence are commensurable by Lemma 2.1.

The proof will make use of certain symmetries of $X(C(p,s))$ (in fact they are symmetries of $(S^3, C(p,s))$). There is a counterclockwise rotation taking each component to the next, which we call α, and a rotation of order two about the circular axis, which we will call β (see Fig.2.4). Let $G(p,s) =< \alpha, \beta >$. When it is convenient, we will

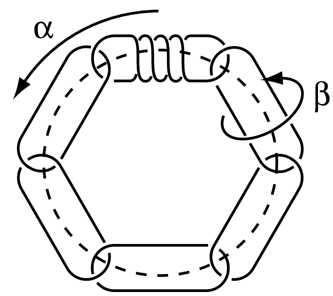

Figure 2.4

assume that $C(p,s)$ is embedded in S^3 so that $G(p,s) \subset U_2$, and

$$\alpha(z_1, z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{-s\pi i}{p}} z_2)$$

$$\beta(z_1, z_2) = (z_1, -z_2)$$

Let $X(W_0)$ be the orbifold 2-fold covered by $X(W)$ pictured in Fig. 2.5. Let $\partial_0 X(W_0)$ be the torus component of $\partial X(W_0)$ (which is covered by $\partial_0 X(W)$) and $X(W_0)(p,s)$ the orbifold obtained from (p,s) Dehn filling on $\partial_0 X(W_0)$ (with respect to m' and l' shown in Fig. 2.5). In the case $p > 0$, this is the quotient of $X(C(p,s))$ by the group $G(p,s)$. To see this, let $V_1 \subset S^3$ be a solid torus containing $C(p,s)$ that is invariant under $G(p,s)$ (The situation for $C(3,2)$ is shown in Fig. 2.6), and let $V_2 = \overline{S^3 \setminus V_1}$ be the complementary solid torus. We note that

$$X(C(p,s))/G(p,s) = (V_1 \setminus N(C(p,s)))/G(p,s) \cup V_2/G(p,s)$$

One easily verifies that $X(W_0) \cong (V_1 \setminus N(C(p,s)))/G(p,s)$ and that $V_2/G(p,s)$ is an orbifold solid torus with core curve having local group $\mathbb{Z}/d\mathbb{Z}$, where $d = |stab(0,z)| = gcd(p,s)$ for any $(0,z) \in S^3$ (we are now considering $G(p,s) \subset U_2$). Therefore, $X(C(p,s))/G(p,s)$ is some

Dehn filling on $\partial_0 X(W_0)$. One can verify that the $(p/d, s/d)$ curve bounds an (orbifold) disk, and hence $X(C(p,s))/G(p,s) \cong X(W_0)(p,s)$, as required. It will therefore suffice to prove that $X(W_0)(p,s)$ and $X(W_0)(p+2s, -p-s)$ cover the same orbifold, since $X(W_0)(p,s) = X(W_0)(-p,-s)$.

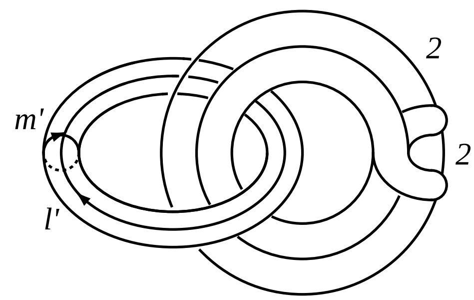

Figure 2.5

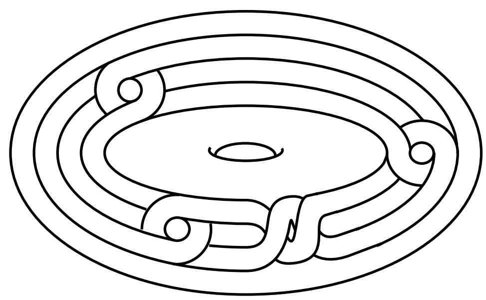

Figure 2.6

We note that if s is even and $gcd(p, \frac{s}{2}) = 1$, $|G(p,s) :< \alpha > | = 2$ and $< \alpha >$ acts freely on $X(C(p,s))$. This fact, and a similar argument to that given in the previous paragraph shows that $X(C(p,s))$ is a p-fold cover of $X(W)(p, \frac{s}{2})$. This proves the second part of Theorem 2.2.

We let $X(W_1)$ be the orbifold shown in Fig. 2.7, which is 2-fold covered by $X(W_0)$, and let $\partial_0 X(W_1) \subset \partial X(W_1)$ be the pillowcase covered by $\partial_0 X(W_0)$. Notice that $X(W_1)$ is the quotient of $X(W_0)$ by an order two rotation τ. As τ extends to a homeomorphism (which we will also call τ) of any Dehn filling on $\partial_0 X(W_0)$, we let $X(W_1)(p,s)$ be the quotient of $X(W_0)(p,s)$ by τ.

We can redraw $X(W_1)$ as in Fig. 2.8. Here we see that $X(W_1)$ admits an order 2 rotational symmetry ρ about a vertical axis. It can be checked that ρ restricted to $\partial_0 X(W_1)$ lifts to a homeomorphism $\widetilde{\rho}$

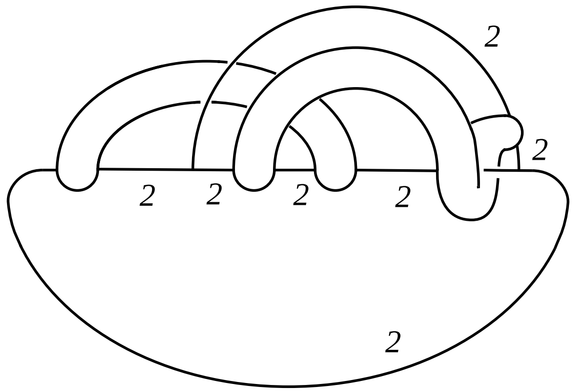

Figure 2.7

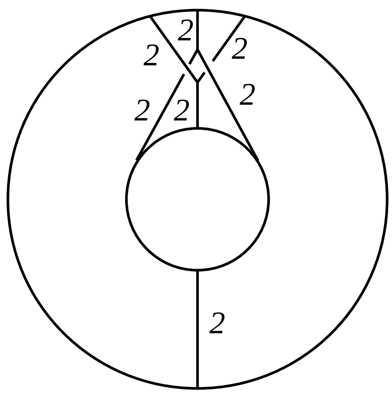

Figure 2.8

of $\partial_0 X(W_0)$ whose induced map on $H_1(\partial_0 X(W_0), \mathbb{Z})$ with respect to m' and l' is given by the matrix:

$$\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

Therefore, given $X(W_0)(p,s)$ and $X(W_0)(p+2s,-p-s)$, we can extend $\widetilde{\rho}$ (which is a map defined only on $\partial_0 X(W_0)$), to a map from the glued in solid torus of $X(W_0)(p,s)$ to that of $X(W_0)(p+2s,-p-s)$. In fact, this can be done so that this extended map commutes with τ. Hence, we can define an (orbifold) homeomorphism of $X(W_1)(p,s)$ onto $X(W_1)(p+2s,-p-s)$ which extends ρ. Thus $X(W_0)(p,s)$ and $X(W_0)(p+2s,-p-s)$ cover the same orbifold, as required.$\square$

We now prove Theorem 2.3 assuming Theorem 4.1.

Theorem 2.3 *$X(W)(p,q)$ is virtually fibered.*

Proof that Theorem 4.1 implies Theorem 2.3. We need to show that for any pair of coprime integers p and q, $X(W)(p,q)$ is virtually fibered. Without loss of generality, we may assume that $p \geq 0$.

If $p > 0$, Theorem 2.2 states that $X(W)(p,q)$ is finitely covered by $X(C(p,2q))$. Since $(p,2q) \neq (2,-1)$, Theorem 4.1 implies $X(C(p,2q))$ is virtually fibered, hence so is $X(W)(p,q)$. If $p = 0$, we must have $q = 1$. The exterior of the Whitehead link is itself fibered [14], and we are filling along the boundary of a fiber. The fibering clearly extends over the filled solid torus and therefore $X(W)(0,1)$ is fibered.□

3 Murasugi Sum

In this section we recall the notion of Murasugi sum (see [3] or [9]) and its applications to constructing fibered links.

Definition 3.1 *We say that the oriented surface R in S^3 with boundary L is the Murasugi sum of the two oriented surfaces R_1 and R_2 with boundaries L_1 and L_2 if there exists a 2-sphere S in S^3 bounding the balls B_1 and B_2 with $R_i \subset B_i$ for $i = 1,2$, such that $R = R_1 \cup R_2$ and $R_1 \cap R_2 = D$ where D is a $2n$-sided disk contained in S (see Fig. 3.1).*

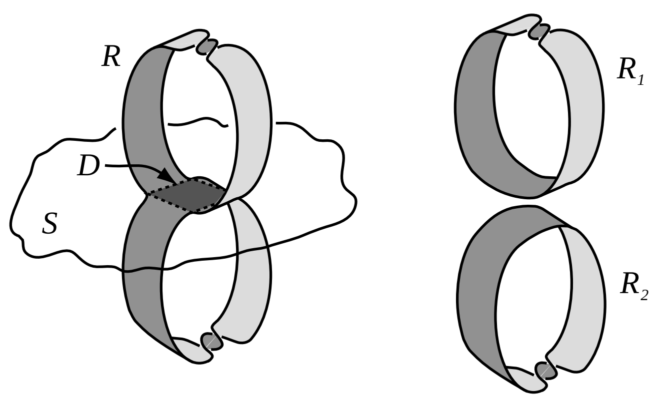

Figure 3.1

The main result concerning the Murasugi sum is the following, due to Gabai [4].

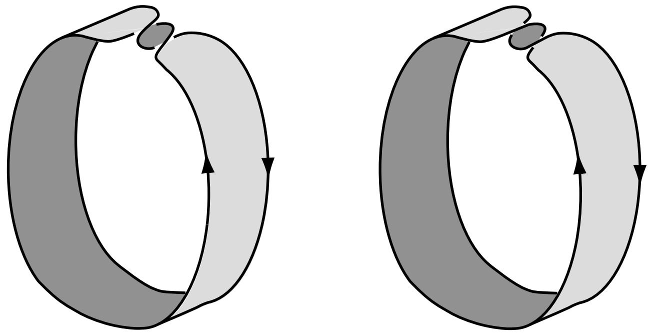

Figure 3.2

Theorem 3.2 *Let $R \subset S^3$, with $L = \partial R$, be a Murasugi sum of oriented surfaces $R_i \subset S^3$, with $L_i = \partial R_i$, for $i = 1, 2$. Then $X(L)$ is fibered with fiber $R \setminus N(L)$ if and only if $X(L_i)$ is fibered with fiber $R_i \setminus N(L_i)$ for $i = 1, 2$.*

Remarks: 1. We will actually only use the "if" direction of Theorem 3.2. This direction that was proven partially by Murasugi [10] and completely by Stallings [16].

2. As an abuse of notation, we will refer to the fiber as R rather than $R \setminus N(L)$. We will also often shorten the statement "$X(\partial R)$ is fibered with fiber R" to "R is a fiber".

The following example of a fibered link exterior in S^3 will be basic to our construction.

Let H_l and H_r be the left and right handed Hopf bands shown in Fig. 3.2 on the left and right respectively. This is an annulus spanning the left and right handed Hopf links, $L_l = \partial H_l$ and $L_r = \partial H_r$. H_l and H_r are both fibers (see [14] for example). Note that by Theorem 3.2, the Murasugi sum of the two Hopf bands pictured in Fig. 3.1 is a fiber in the fibering of the exterior of the figure 8 knot.

4 Virtually Fibering Chain Links

In this section we prove the following

Theorem 4.1 *$X(C(p, s))$ is virtually fibered if and only if $(p, s) \neq (2, -1)$.*

We begin with some preliminary lemmas.

Lemma 4.2 *$X(C(p,s))$ is fibered if $p \geq -s \geq 0$ and $(p,s) \neq (2,-1)$.*

Proof. When $p = 1$ we have $s = 0$ or $s = -1$. In this case, $C(p,s)$ is the unknot, and its exterior is fibered. So suppose $p \geq 2$. Since $p \geq -s \geq 0$ (and $(p,s) \neq (2,-1)$), we note that $C(p,s)$ has a projection given by Fig. 4.1 (a) ($s \leq -2$) or Fig. 4.1 (b) ($s > -2$). To see this we proceed as follows.

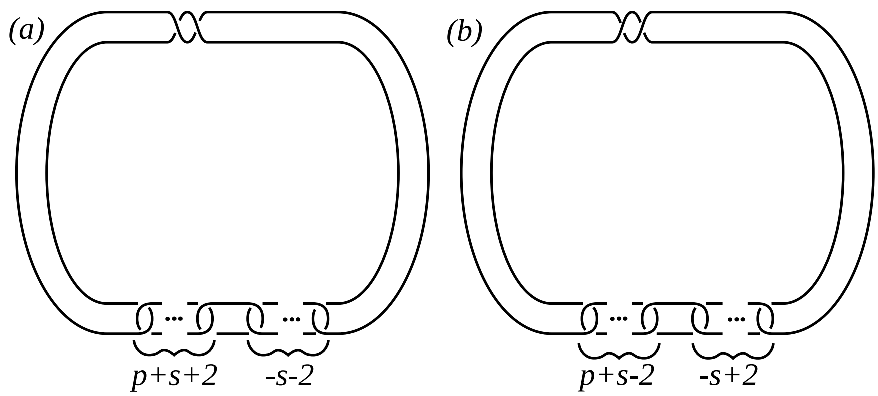

Figure 4.1

In the case $s \leq -2$, we can remove a crossing at a right-handed half twist by changing the crossings at one of the clasps (see Fig. 4.2). We do this for all but 2 of the $-s$ crossings, which is possible since $p \geq -s$. When $s = -1$, we can similarly change the single crossing and add a crossing by changing the crossings at three of the clasps (this is where we need $(p,s) \neq (2,-1)$). When $s = 0$, we can add two crossings by changing the crossings at two of the clasps.

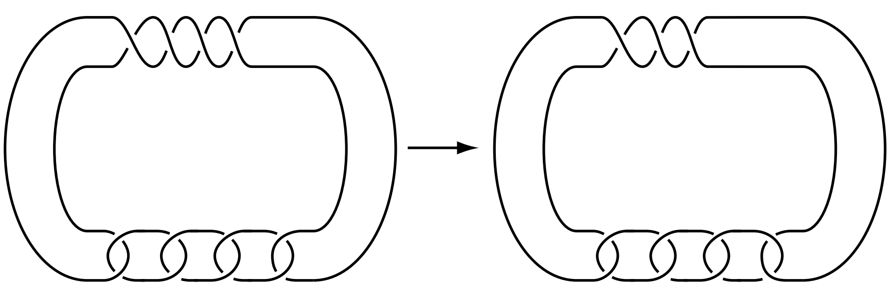

Figure 4.2

Now orient the link and perform Seifert's algorithm to obtain the Seifert surface $R(p,s)$ shown in Fig. 4.3 (a)($s \leq -2$) and Fig. 4.3 (b) ($s > -2$). We claim that this surface is a fiber in a fibering of $X(C(p,s))$. We provide the details for the case $s \leq -2$. The case $s > -2$ follows similarly.

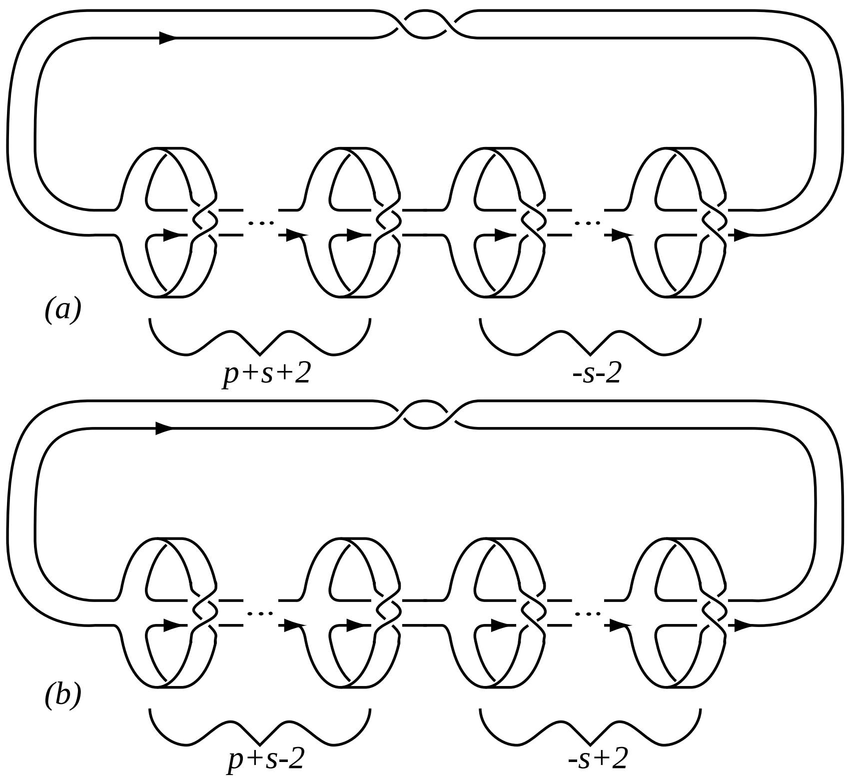

Figure 4.3

We proceed by induction on p. Theorem 3.2 implies $R(2,-2)$ is a fiber since it is the result of a Murasugi sum of 3 copies of H_l. Suppose now that $p > 2$, and that for all $k < p$ and $2 \leq -l \leq k$, $R(k,l)$ is a fiber. To see that $R(p,s)$ is a fiber, there are two cases to consider; $-s = p$ and $-s < p$. For $-s = p$, $R(p,s)$ is a Murasugi sum of $R(p-1,s+1)$ and H_r (see Fig. 4.4(a)), and for $-s < p$, it is a Murasugi sum of $R(p-1,s)$ and H_l (see Fig. 4.4(b)). By the induction hypothesis and Theorem 3.2, $R(p,s)$ is a fiber. This proves the claim for $s \leq -2$.□

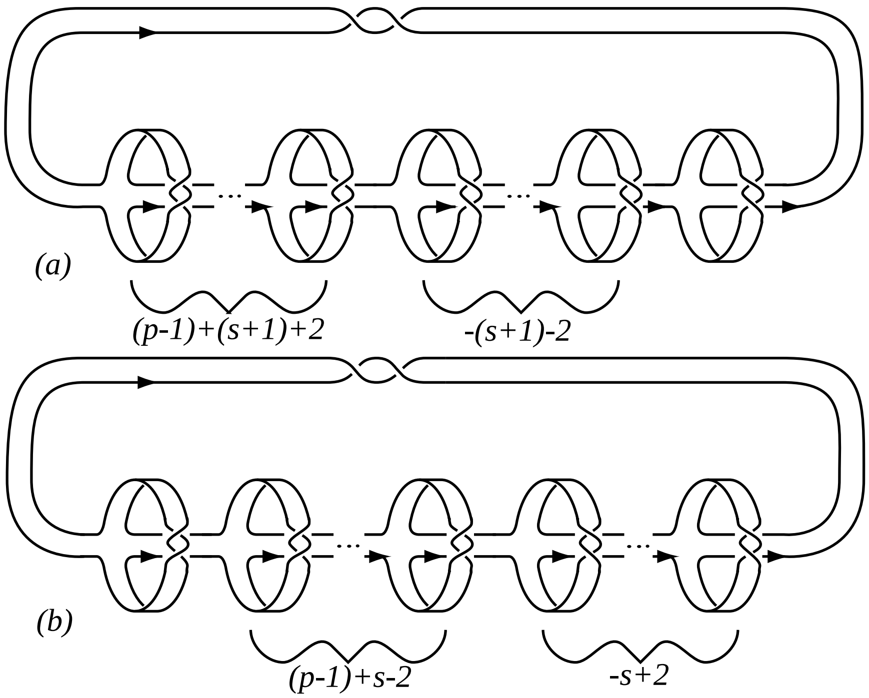

Figure 4.4

Remark: Notice that the Lemma is not as strong as it could be. It is easy to make the argument work for $p + 2 \geq -s \geq -2$. However, we will only need the lemma as stated.

Lemma 4.3 *Given* $p > 0$ *and* $s \leq 0$*,* $X(C(p, s))$ *is virtually fibered if and only if* $(p, s) \neq (2, -1)$*.*

Proof. If $(p, s) = (2, -1)$, then $C(p, s)$ is the 2 component unlink. $X(C(p, s))$ is therefore a reducible orientable compact 3-manifold, and the only such manifolds which are virtually fibered are virtual sphere bundles, which are closed 3-manifolds. Hence $X(C(p, s))$ is not virtually fibered. This proves the "only if" part.

To prove the other implication we note that by Lemma 4.2 we may assume that $-s > p$. Since $p \geq 1$ this implies $s \leq -2$. Let

$$p' = -(p + 2s)$$

$$s' = p + s$$

By our assumption, we have

$$s' = p + s < 0$$

and

$$p' = -p - 2s > s - 2s = -s \geq 2$$

This implies $p' \geq 3$ (in particular $(p', s') \neq (2, -1)$). We also have

$$p' = -(p + 2s) = -(p + s) - s = -s' - s > -s'$$

The hypotheses of Lemma 4.2 are therefore satisfied by (p', s'), whence $X(C(p', s'))$ is fibered. Furthermore, by Theorem 2.2, $X(C(p, s))$ and $X(C(p', s'))$ are commensurable. Thus $X(C(p, s))$ is virtually fibered.□

Proof of Theorem 4.1. By the previous two lemmas, we need only show that if $s > 0$, then $X(C(p, s))$ is virtually fibered. In this case, we will show that there is an orientation reversing homeomorphism of $X(C(p, s))$ onto $X(C(p, -s-p))$. We take the reflection of $C(p, s)$ (i.e. $C(p, s)$ with the crossings changed). This is an orientation reversing homeomorphism of S^3. Note that the s left handed half twist, have become s right handed half twists. Using the same idea as in the proof of Lemma 4.2, we can add p more right handed half twists by changing the crossings on each of the clasps. This gives us $C(p, -s - p)$, and the orientation reversing homeomorphism of S^3 maps $X(C(p, s))$ onto $X(C(p, -s - p))$.

Now, since $s > 0$ and $p > 0$, we know that $-s-p < -1$. Therefore, by Lemma 4.3, $X(C(p, -s - p)) \cong X(C(p, s))$ is virtually fibered.□

5 Applications

5.1 Twist knots

As mentioned in the introduction, we can use Theorem 2.3 (or Theorem 4.1) to prove

Corollary 5.1 *$X(K_k)$ is virtually fibered for all $k \geq 0$. Moreover, for $k > 2$, $X(K_k)$ is hyperbolic and not fibered.*

Remark: For the cases $k = 0, 1, 2$ we have the unknot, trefoil, and figure 8, respectively, each of which is fibered.

Proof. To prove the first statement we simply note that K_k is the chain link $C(1,k)$.

To prove the second statement, we first note that the Alexander polynomial of a knot with fibered exterior is monic (see [14]). A calculation shows that the Alexander polynomial of K_k is

$$-\frac{k}{2}t^2 + (k+1)t - \frac{k}{2}$$

when k is even, and

$$\frac{k+1}{2}t^2 - kt + \frac{k+1}{2}$$

when k odd. Thus $X(K_k)$ is not fibered if $k > 2$.

We can also consider $X(K_k)$ as a result of Dehn filling on one component of the boundary of the exterior of the Whitehead link, $X(W)$. With the same framing as in Section 2, $X(K_k)$ is homeomorphic to $X(W)(1, \frac{k}{2})$ when k is even, and $X(W)(1, \frac{-(k+1)}{2})$ when k is odd (note that the homeomorphism reverses orientations when k is odd). Since $X(W)$ is hyperbolic, Thurston's Dehn Surgery Theorem implies that all but a finite number of the $X(K_k)$ are hyperbolic (see [17]). In fact in [11], the hyperbolic Dehn surgery space of one component of $\partial X(W)$ is calculated, and is shown to contain all but 6 slopes (namely $(0,1)$,$(1,0)$,$(1,-1)$,$(2,-1)$,$(3,-1)$, and $(4,-1)$), from which it follows that $X(K_k)$ is hyperbolic if $k \geq 2$.□

5.2 The Knot 7_4

As another consequence of Theorem 2.3 we record

Corollary 5.2 *The exterior of the knot 7_4 is not fibered but is virtually fibered.*

Proof. The knot 7_4 from the knot tables [14] is pictured in Fig. 5.1. Its Alexander polynomial is

$$4t^2 - 7t + 4$$

Hence 7_4 is not fibered.

To show that $X(7_4)$ is virtually fibered, we will show that it is commensurable with $X(W)(-3,2)$. We use the same notation as in the proof of Theorem 2.2.

Since the $(-3,4)$ curve on $\partial_0 X(W_0)$ lifts to the $(-3,2)$ curve on $\partial_0 X(W)$, $X(W_0)(-3,4)$ is covered by $X(W)(-3,2)$. Let $X(W_1)(-3,4)$ be the quotient of $X(W_0)(-3,4)$ by τ. $X(W_1)(-3,4)$ is shown in Fig. 5.2 (this a 3-ball with the singular locus as indicated). This can be redrawn as in Fig. 5.3.

We see that $X(7_4)$ is a 4-fold cover of the orbifold in Fig. 5.4 which we call $X(7_4)'$. It is the quotient by the $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ symmetry group generated by order two rotations about vertical and horizontal axes. We can redraw $X(7_4)'$ as in Fig. 5.5. From this picture one observes that $X(7_4)'$ admits an order two rotation about an axis which runs "perpendicular to the paper". This gives the quotient $X(7_4)''$ pictured in Fig. 5.6. We redraw $X(7_4)''$ as in Fig. 5.7, and observe that it is homeomorphic to $X(W_1)(-3,4)$.□

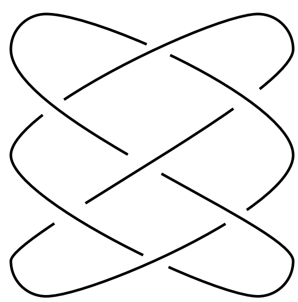

Figure 5.1

5.3 Finitely Generated Intersection Property

We briefly describe one final consequence of Theorem 2.3.

Recall that a group has the *finitely generated intersection property* (FGIP) if for every two finitely generated subgroups $H, K < G$, $H \cap K$ is also finitely generated. In [7] Jaco shows that if a compact 3-manifold M is virtually fibered with fiber F satisfying $\chi(F) < 0$, then $\pi_1(M)$ does not have FGIP.

Corollary 5.3 *$\pi_1(X(W)(p,q))$ has FGIP if and only if $(p,q) = (1,0)$.*

Proof. As was mentioned in the proof of Corollary 5.1, all but the six surgeries with slope $(0,1)$,$(1,0)$,$(1,-1)$,$(2,-1)$,$(3,-1)$, and $(4,-1)$, are hyperbolic [11], so the virtual fibers must have negative Euler characteristic. Theorem 2.2 implies $X(W)(3,1)$ and $X(W)(4,1)$ have

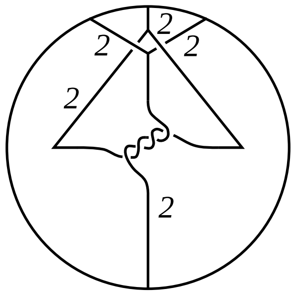

Figure 5.2

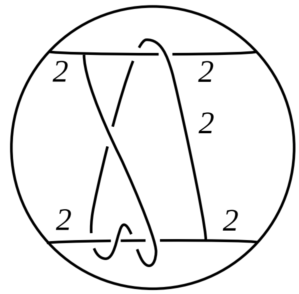

Figure 5.3

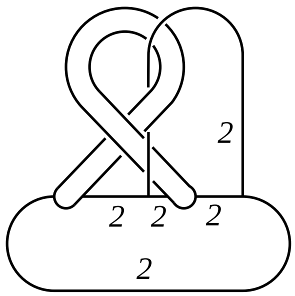

Figure 5.4

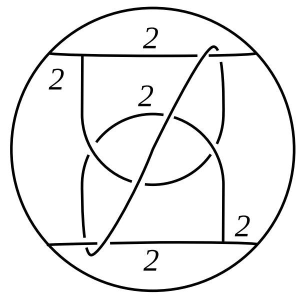

Figure 5.5

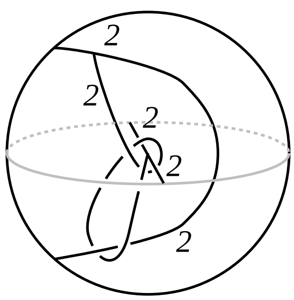

Figure 5.6

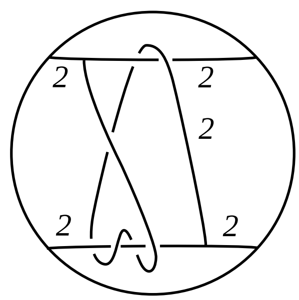

Figure 5.7

covers with at least 3 boundary components, so that any virtual fiber F must have $\chi(F) < 0$. Similarly, one can show that $X(W)(2,-1)$ has a cover with 3 boundary components. $X(W)(0,1)$ and $X(W)(1,-1)$ are punctured torus bundles. $X(W)(1,0)$ is a solid torus.$\square$

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