

## HOMEWORK 10 – SOLUTIONS

### PROBLEM 1

(a). Fix a model  $\mathfrak{M}$ , let  $T$  be  $Th(\mathfrak{M})$ . Pick an element,  $m$ , of the model. Show that the set of all formulas that hold of  $m$  form a type.

*Proof.* Let  $p = \{\varphi_i\}$  be the set of formulas true of  $m \in \mathfrak{M}$ . We want to show that this set of formulas is consistent. Take some finite set  $\varphi_{n_0}, \dots, \varphi_{n_k}$  of formulas from  $p$ . We need to show that  $Th(\mathfrak{M})$  proves  $\exists x(\varphi_{n_0}(x) \wedge \dots \wedge \varphi_{n_k}(x))$ . But this is clear since  $\varphi_{n_0}(m) \wedge \dots \wedge \varphi_{n_k}(m)$  □

(b). Now let  $\mathfrak{N}$  be  $(\mathbb{N}, +, \cdot, <, 0, 1)$ . Is every type realized by some element of  $\mathbb{N}$ ? (An element  $n$  *realizes* a type iff every formula in the type is true of  $n$ ).

*Proof.* The answer is no, and this can be seen by considering the type that says consists of the formulas  $x > n$  for each  $n$  in  $\mathbb{N}$ . First we must check to see that this set of formulas really is a type. That is, we must prove that it is consistent. But this is not difficult, since any finite set of formulas from the type only mentions finitely many natural numbers. Let  $m$  be the largest natural number mentioned, and let  $x = m + 1$ .

It is similarly clear that  $p$  is not realized in  $\mathbb{N}$  as there is no single  $k \in \mathbb{N}$  greater than each  $n$ . □

(c). Keeping  $T$  and  $L$  as in part (b), is every type consisting of only finitely many formulas realized in  $\mathbb{N}$ ?

*Proof.* Here the answer is “yes”. Consider a type consisting of finitely many formulas, say  $\psi_1(x), \dots, \psi_n(x)$ .  $T$  proves  $\exists x(\psi_1(x) \wedge \dots \wedge \psi_n(x))$ . Thus there must be some  $m$  such that  $\psi_1(m) \wedge \dots \wedge \psi_n(m)$ . Such an  $m$  realizes the type. □

(d). Again keeping  $T$  and  $L$  as in part (b), is every type realized in some model  $\mathfrak{N}$  that is elementarily equivalent to  $\mathbb{N}$ ?

*Proof.* Again the answer is “yes”. The proof uses the compactness theorem. Let  $p$  be a type. Say  $p := \{\varphi_i(x)\}$ . Expand  $L$  to  $L'$  by adding a new constant  $c$ . Let  $T' := T \cup \{\varphi_i(c) \mid \varphi_i(x) \in p\}$ . Suppose that we knew that  $T'$  had a model,  $\mathfrak{N}$ . Then, considering this model in the language  $L$ , (that is, forgetting the constant  $c$ ) we see that  $\mathfrak{N}$  is elementarily equivalent to  $\mathbb{N}$  (since both satisfy  $T$ ). Furthermore, whichever element of  $\mathfrak{N}$  corresponded to the constant  $c$  has to realize  $p$ .

Thus it remains to show that every finite subset of  $T'$  has a model. Each finite subset,  $T_F$ , of  $T'$  mentions only finitely many sentences of the form  $\varphi_{n_0}(c), \dots, \varphi_{n_k}(c)$ . Since  $p$  is a consistent set of formulas,  $T$  proves  $\exists x(\varphi_{n_0}(x) \wedge \dots \wedge \varphi_{n_k}(x))$ . Thus some  $m_F \in \mathbb{N}$  is such that  $\varphi_{n_0}(m_F) \wedge \dots \wedge \varphi_{n_k}(m_F)$ . Let  $\mathfrak{N}_F$  be the model identical to the  $\mathbb{N}$  except with the new constant,  $c$ , interpreted as the element  $m_F$ . Then it is clear that  $\mathfrak{N}_F \models T_F$ .

Thus since every finite subset of  $T'$  has a model, we apply the compactness theorem to conclude that that  $T'$  itself has a model, and we are done.

□

## PROBLEM 2

Show that for any language  $L$  complicated enough that the valid sentences in  $L$  are undecidable, that the satisfiable sentences in  $L$  are not even enumerable.

*Proof.* Note that the set of sentences that are never true in any model is an enumerable set: just modify the program that outputs the valid sentences, and modify it to output  $\neg\phi$  whenever it would output  $\phi$ .

On the other hand, the set of sentences that are never true cannot be decidable. If it were, you could design a program to decide the set of valid sentences (since a sentence  $\phi$  is valid iff  $\neg\phi$  is never true).

Thus the complement of the set of sentences that are never true is not enumerable. But this is the set of satisfiable sentences.

□

## PROBLEM 3

Let  $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$ . Show that any set defined by a formula,  $\psi(z)$ , of the form  $\exists x_1, \dots, \exists x_n \varphi(x_1, \dots, x_n, z)$  where  $\varphi$  has no quantifiers, is enumerable.

*Proof.* First we show that any set defined by  $\varphi(\vec{x})$ , where  $\varphi(\vec{x})$  is quantifier free is decidable. This is induction on the complexity of  $\varphi$ . We need to show that there is a register machine that given an assignment,  $\beta$ , of  $\vec{x}$  to a tuple of natural numbers  $\vec{n}$ , can determine whether or not  $((\mathbb{N}, +, \cdot, <, 0, 1), \beta) \models \varphi$ . (As this is the definition of  $\vec{n}$  being in the set defined by  $\varphi$ .)

First suppose that  $\varphi$  is atomic. That is,  $\varphi$  is of the form  $t_1 < t_2$  or  $t_1 = t_2$  for  $t_1, t_2$  terms. But it is easy to see that there is a register machine that can calculate  $\beta(t_1)$  and  $\beta(t_2)$  and then check whether or not these two numbers equal to each other or if one is greater than the other.

Now suppose  $\varphi$  is of the form  $\theta_1(\vec{x}) \wedge \theta_2(\vec{x})$ . Then the set defined by  $\varphi$  is just the intersection of the sets defined by  $\theta_1$  and  $\theta_2$ . And we have shown that the intersection of two decidable sets is decidable.

Now suppose that  $\varphi$  is of the form  $\neg\theta(\vec{x})$ . Then the set defined by  $\varphi$  is the complement of the set defined by  $\theta$  and the complement of a decidable set is decidable.

Thus, induction shows that if  $\varphi$  is quantifier free, then the set it defines is decidable.

Now consider  $\psi$ . Let  $P$  be as follows. On input  $n$ ,  $P$  wants to know whether  $\psi(n)$  holds. So,  $P$  goes through all tuples  $\vec{m}$  in lexicographic order, and tests whether or not  $\varphi(\vec{m}, n)$  holds. If  $P$  finds  $\vec{m}$  which works, then  $\psi(n)$  halts, and  $P$  halts and says “yes”.

□