HOMEWORK 11

Due Wednesday, May 3rd

Problem 1

Show that the valid sentences of first order logic, for any language L, are decidable by an oracle register machine with oracle Π_{halt}.

Proof. First recall that the set of valid sentences in first order logic are enumerable. Let us see why this is true in more detail. One may create a program P that runs as follows: First P takes a string w as input. It checks to see that w is a well formed sentence in first order logic. If it is not, it goes into an infinite loop. If w is a well formed sentence, it begins running through all possible proofs using the sequent calculus in lexiographic order. If it finds a proof of the formula w it stops and outputs "yes". Otherwise it keeps running.

Clearly P halts if and only if w is a valid sentence. Thus P shows that the collection of valid sentences is enumerable.

Now consider the following program Q that uses Π_{halt} as an oracle. On input w, Q writes the source code for a program, P_w, that acts as follows: P_w loads w into memory, and then runs as P from the previous paragraph. Now Q asks whether the program P_w halts. If the answer is "yes" then w is a valid sentence. If the answer is "no", it is not. Thus Q shows that the set of valid sentences is decidable using Π_{halt} as an oracle. □

Problem 2

(a). Given L any language, show that any formula $\psi(x)$ is equivalent to one with all the quantifiers at the beginning.

(b). Do this for $L = \{+, \cdot, 0, 1, <\}$ and ψ being the formula

$$(\neg\forall x(x < y^2)) \to (\exists x(x < z^3))$$

Problem 3

Here is a definition (which I will also talk about in class on Friday): Let t be a term in the language $L = \{+, \cdot, <, 0, 1\}$. Say that a quantifier is *bounded* in a formula if it appears in the form $\exists x(x < t) \wedge \ldots$ or $\forall x(x < t) \to \ldots$. That is, a quantifier is bounded if the variable being quantified over ranges only over things less than t.

(a). Let $\psi(x_1, \ldots, x_k)$ be a formula with of the form

$$Qy_1(Qy_2(\ldots(Qy_n\varphi(y_1, \ldots, y_n, x_1, \ldots, x_k)\ldots)$$

where each Qy_i stands for a bounded quantifier, and where φ contains no quantifiers. That is Qy_i is either $\exists y_i(y_i < t_i) \wedge \ldots$ or $\forall y_i(y_i < t_i) \to \ldots$ for some term t_i. Prove that the subset of the $\mathbb{N}^k$ defined by $\psi(x)$ is decidable.

(b). Let $\theta(x)$ be of the form $\exists z_1 \exists z_2 \exists z_3 \ldots \psi(z_1, \ldots z_n, x)$ for ψ as in part (a) (that is, ψ has only bounded quantifiers followed by something quantifier free). Show that the subset of $\mathbb{N}$ defined by $\theta(x)$ is enumerable.