HOMEWORK 1: SOLUTIONS

PROBLEM 1

Prove that the composition of two bijections is again a bijection.

Proof. Let $f : A \to B$ and $g : B \to C$ be bijections. Take a_1, a_2 in A with $a_1 \neq a_2$. Since f is an injection, $f(a_1) \neq f(a_2)$. Since g is an injection, $g(f(a_1)) \neq g(f(a_2))$ and hence $g \circ f$ is an injection. Now take $c \in C$. Since g is a surjection there is some b that maps to c. Since f is a surjection there is some $a \in A$ that maps to b. Thus $g \circ f$ maps a to c, and is a surjection. Since it is both injection and a surjection, it is a bijection. $\square$

PROBLEM 2

Prove directly from the definitions that the union of two countable sets is countable. Feel free to use without proof the fact that the union of two finite sets is finite.

Proof. Recall that a set A is countable iff there is an injection from B to $\mathbb{N}$. Suppose $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ are injections witnessing that A and B are countable. Now let $C := B - A$. Define a map from $A \cup B$ to C as follows. For $a \in A$ let $h(a) := 2 \cdot f(a)$ and for b in C let $h(b) := 2 \cdot g(b) - 1$. Since A is mapped into the even numbers and that part of B which isn't in A is mapped to the odd numbers, it is easy to see that h is an injection. $\square$

PROBLEM 3

Prove that the set of all finite subsets of $\mathbb{N}$ is countable.

Proof. Probably the easiest way to do this problem is to show first that the set of all subsets of $\mathbb{N}$ of size k is countable, and then to show that the countable union of countable sets is countable.

Note that to show that a set A is countable it suffices to give a injection from A to B where B is countable. Thus the map that assigns the finite set $\{n_1, \ldots, n_k\}$ to the tuple $(n_1, \ldots, n_k)$ shows that the set of all subsets of $\mathbb{N}$ of size k is countable, since we proved in class that $\mathbb{N}^k$ is countable.

Now consider $(A_i)_{i \in \mathbb{N}}$, and suppose that each A_i is countable. To make things easier, we will replace the collection of A_i with another collection $(B_i)_{i \in \mathbb{N}}$ such that the B_i are pairwise disjoint. Let $B_0 := A_0$. Let $B_i := A_i - (A_0 \cup \ldots \cup A_{i-1})$. Now each B_i is countable, and $\bigcup A_i = \bigcup B_i$. For each B_i let $f_i : B_i \to \mathbb{N}$ be an injection. Define $f : \bigcup B_i \to \mathbb{N}^2$ as follows: map an element $b \in B_i$ to $(i, f_i(b))$. It is easy to see that f is an injection, and thus shows that $\bigcup A_i = \bigcup B_i$ is countable. $\square$

PROBLEM 4

Prove DeMorgan's Laws: that is, prove that $(A \cup B)^C = A^C \cap B^C$ and that $(A \cap B)^C = A^C \cup B^C$

Proof. Note that $x \in (A \cup B)^C$ iff x is not in $(A \cup B)$ iff x is in neither A nor B iff x is not in A and x is not in B iff x is in A^C and x is in B^C. The proof that $(A \cap B)^C = A^C \cup B^C$ is similar. $\square$

PROBLEM 5

Here's a puzzle to let you practice your mathematical induction skills. Consider the following game: There are two piles of coins, and two players. The players alternate turns, and on each turn a player can take any positive number of coins from one of the two piles. The player that takes the last coin wins. The two piles begin with the same number of coins. Does one of the players have a winning strategy? Prove it.

Proof. Player 2 has a winning strategy, namely taking the exact same number of coins that Player 1 took, but from the other pile. Here's a proof by induction. Suppose that there is one coin in each pile. Then Player 1 has to take one coin and Player 2 takes the other. Now suppose that Player 2 has a winning strategy when both piles have 1, 2, 3, ... or n many coins. We consider the case when each pile has $n+1$ coins. Then Player 1 takes k coins from one pile and Player 2 take k coins from the other pile. Now we are in a situation where Player 1 is about to go first and each pile has $n-k$ coins. By induction, Player 2 has a winning strategy in this situation. $\square$

PROBLEM 6

Prove that each subset of a countable set is countable.

Proof. As in class, we reduce to the case where A is an infinite subset of $\mathbb{N}$ and we want to show that there is a bijection between A and $\mathbb{N}$. We build the function that I described in class. We want to show that it is injective and surjective.

First the proof that it is injective. If f were not injective, then $f(m) = f(n)$ for some $m \neq n$. Say that $m < n$. Then the restriction of f to $\{1, \ldots, n\}$ is already not an injective function. Thus, to show that f is injective, it suffices to show that the restriction of f to $\{1, \ldots, n\}$ is injective for all n, and we will do this by induction.

Clearly, the restriction of f to $\{1\}$ is injective.

Now assume that f restricted to $\{1, \ldots, k-1\}$ is injective. We want to show that $f(k)$ is not equal to any of $f(1), \ldots, f(k-1)$. But this is clear, since $f(k)$ is chosen to be an element of $A \setminus \{f(1), \ldots, f(k-1)\}$.

Second we must show that f is surjective. This is a bit harder. We want to find a statement that we can prove by induction (so it should be about the function restricted to some initial segment of the natural numbers), but that should imply surjectivity, which is a property of the function as a whole.

Suppose that we knew that f were an increasing function. Then we would be able to check surjectivity of f on an initial segment of the natural numbers. For suppose that f were not surjective. Then there would be some $n \in A$ such that n is not in the image of f. But if f is increasing, if n is not in the image of f restricted to $\{1, \ldots n\}$ then it will never be in the image. Conversely, if n is in the image of an increasing f, it is already in the image of f restricted to $\{1, \ldots, n\}$.

Thus, we may prove that f is surjective in two steps: first show that f is increasing, and second show that $A \cap \{1, \ldots, n\}$ is contained in the image of f restricted to $\{1, \ldots, n\}$. Since f is increasing iff f restricted to $\{1, \ldots, n\}$ is increasing for each n, we see that both of these statements are of the sort that look like they can be proven by induction.

We could prove both of these statement with separate inductive arguments, but there's no harm in proving them both at once, either, and this will save typing.

Note that $f(1)$ can be no smaller than 1 since $A \subseteq \mathbb{N}$. So f restricted to $\{1\}$ is increasing. Furthermore, we set $f(1)$ to be the least element in A. Thus if 1 is in A, $f(1) = 1$, and we have confirmed that $A \cap \{1\}$ is contained in $f(\{1\})$.

Now assume that for $j \in \{1, \ldots, k-1\}$, $j \leq f(j)$, and that $A \cap \{1, \ldots, k-1\}$ is contained in $f(\{1, \ldots, k-1\})$. Since we choose $f(k)$ to be the least element of $A \setminus \{f(1), \ldots, f(k)\}$, we see that $f(k)$ is at least $f(k-1)+1$. Thus $k-1 \leq f(k-1)$ implies that $k = (k-1) + 1 \leq f(k-1) + 1 \leq f(k)$. It remains only to check that if k is in A, then k is in $f(\{1, \ldots, k\})$. But if k is in A, then either it is already in $f(\{1, \ldots, k-1\})$, or since each element of A less than k is in $f(\{1, \ldots, k-1\})$, k must be the least element of $A \setminus f(\{1, \ldots, k-1\})$, in which case $k = f(k)$.

Thus we have shown that f is surjective in addition to being injective, and we are done. $\square$