HOMEWORK 2

Problem 1

Let A and B be sets. Prove that there is a surjection from A to B iff there is an injection from B to A.

Proof. First suppose that there is a surjection $f : A \to B$. Thus for any $b \in B$ there is some collection of elements $f^{-1}\{b\}$ such that for any a in $f^{-1}\{b\}$, a is mapped to b by f. Pick any such a and set $g(b) := a$. Do this for each $b \in B$. Now it is easy to see that g is an injection. Suppose that $g(b_1) = g(b_2) = a$. Then $f(a) = b_1$ and $f(a) = b_2$, and so $b_1 = b_2$.

A brief aside before moving on to the second half of the proof. Some of you may be wondering how I can use the symbol f^{-1} when f is not a bijection. Let me remind you that the notation f^{-1} has two similar but not identical meanings. First when one writes $f^{-1}(B)$ and B is a subset of the range of f, then $f^{-1}(B)$ is the set of elements that f maps to B. Second, when one writes $f^{-1}(b)$ and b is an element of the range of f, then this means "the inverse function of f applied to the element b", and, of course, this only makes sense when f is a bijection.

Now to prove the other direction. Suppose that there is an injection $g : B \to A$. Let $A_0 \subseteq A$ be the image of B under g. Then g defines a bijection $g : B \to A_0$. Now define $f : A \to B$ as follows. Pick some b_0 in B. Let f map everything in A that's not in A_0 to b_0, and for $a \in A_0$, let $f(a) := g^{-1}(a)$. It is clear that this is a surjection, since for any $b \in B$, f maps $g(b)$ to b.

□

Problem 2

Prove that the set of bijections from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

Proof. Let S be the set of bijections from $\mathbb{N}$ to $\mathbb{N}$.

There are several ways to do this proof, but the two easiest are to imitate the proof from class that the reals were uncountable, or else to find a injection from an uncountable set into the set of bijections from $\mathbb{N}$ to $\mathbb{N}$.

I pointed out that these were the two things to try to those of you that came to my office hours for help on this question, but I also hinted that imitating the proof that the reals are uncountable might be easier. Now that I am writing up solutions, however, I think that finding an injection of the interval $(0, 1)$ into the set of bijections is much easier than I thought it was. Anyhow, I apologize if you did more work than you had to.

Now lets define $f : (0, 1) \to S$. First note that we can write any element $b \in (0, 1)$ in binary, that is, as $b = 0.b_1b_2b_3\ldots$ with each b_i equal to either 0 or 1. Now define a function from $f_b : \mathbb{N} \to \mathbb{N}$ as follows: If $b_1 = 1$, then f_b will switch 1 and 2. If $b_1 = 0$ then f_b sends 1 to 1 and 2 to 2. Likewise, if $b_k = 1$, f_b will switch $2k - 1$ and $2k$. Otherwise, f_b keeps these constant. So if b is $\frac{3}{16}$ then $b = 0.0011000\ldots$, so $f(4) = 5$, $f(5) = 4$, $f(6) = 7$, $f(7) = 6$, and for all other $n \in \mathbb{N}$, $f(n) = n$.

It should be clear that f_b is a permutation of $\mathbb{N}$. It remains to show that the map that sends $b \mapsto f_b$ is an injection. Take a, b distinct elements of $(0,1)$. Then, if $a = a_1a_2a_3\dots$ and $b = b_1b_2b_3\dots$ then for some k, $a_k \neq b_k$. Thus f_a and f_b do different things to $2k$, and are thus different functions.

Now I'll quickly sketch a second proof, one by diagonalization. Suppose that you have a surjection from $\mathbb{N}$ to set of bijections from $\mathbb{N} \to \mathbb{N}$. We will think of a bijection from $\mathbb{N} \to \mathbb{N}$ as a sequence $(n_1, n_2, n_3, \dots)$ where each number appears precisely onces in the sequence.

Suppose our "surjection" maps k to $f(k) := (n_{1,k}, n_{2,k}, n_{3,k} \dots)$. We want to find a sequence that contains each natural number precisely once and that's not in the image of our supposed surjection.

At stage 1, we set $m_1 := 1$ unless $n_{1,1} = 1$, in which case, we set $m_1 := 2$ and $m_2 := 1$. Thus at the end of stage 1, we have insured that the sequence that we are building is not equal to $f(1)$, and we have specified no more than the first two elements of the sequence.

Now assume we have completed stages 1 to $i-1$, and have decided on no more than the first $2(i-1)$ element of the sequence, and have insured that the sequence is not equal to $f(1), \dots f(i-1)$. We describe stage i: Suppose that we have decided on precisely the first l things in our sequence. We look at the $(l+1)th$ number in the sequence $f(i)$. We set m_{l+1} to be the least natural number not mentioned yet used in our sequence unless $n_{l+1,i}$ is this number. In this case we set m_l to be one more than this number, and set m_{l+2} to be this number. Thus we have insured that we are building will not be equal to $f(i)$, we have used no more than the first $2i$ numbers.

It remains to show that the sequence constructed contains every natural number exactly once, but if one examines the proof, one will see that at stage i, if one has used l numbers, then one has used each number from 1 to l precisely once.

□

Problem 3

Prove that $|[0,1]| = |\mathbb{R}|$.

Proof. We need to give an injection of $[0,1]$ into $\mathbb{R}$ and another injection from R into $[0,1]$. The first is obvious (inclusion); the second requires a bit of thought. Consider the function $\cot(x)$. Note that this gives a bijection from the interval $(0,\pi)$ to the $\mathbb{R}$. Thus $f(x) := \cot^{-1}(x)$ gives a bijection from $\mathbb{R}$ to $(0,\pi)$. Thus $g(x) := f(x)/\pi$ is a bijection $\mathbb{R} \to (0,1)$ (and hence an injection $\mathbb{R} \to [0,1]$). □

Problem 4

Prove that every formula in propositional logic has an even number of parentheses.

Proof. This wasn't meant to be a difficult question. It was just meant to let you practice doing induction on the length or rank of formulas.

First the base case. A formula, ϕ, has rank 1 iff it is a propositional letter (such as p, q, r, etc.). Note that each of these has zero parentheses, and zero is even.

Now assume that we have proven the claim when ϕ has rank $1, \dots, k-1$, and consider the case of rank $\phi = k$.

Case 1: ϕ has the form $(\theta_1 \rightarrow \theta_2)$. Then the rank of θ_1 and $theta_2$ are both less than the rank of ϕ. That is, they both have rank less than k and so the inductive hypothesis applies to them. Thus θ_1 and $theta_2$ both have an even number of parentheses. Thus the number of parentheses of ϕ is an even number plus and even number plus two, which is again an even number.

Now one needs to do cases for ϕ having the form $(\theta_1 \leftrightarrow \theta_2)$, $(\theta_1 \wedge \theta_2)$, $(\theta_1 \vee \theta_2)$, and $\neg\theta$. All of these work just like the first case.

□

Problem 5

Which of the following are tautologies? Prove.

(a). $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$

(b). $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

(c). $(p \wedge q) \rightarrow (p \vee q)$

Proof. To do this, one writes down the truth tables for each one, and checks whether or not the final column is all "T's". After trying to get latex (the typesetting program that I use to make the solutions) to produce nicely formatted truth tables for the last hour or so, I'm giving up. If you want to see how to do this problem, ask me after class or in office hours. □

Problem 6

A king has a prisoner locked up in a tower. The tower has two exits, one of which leads to freedom, the other into the torture chamber. A guard is standing in front of each exit. One of the guards always tells the truth, the other always lies. The guards each know what is behind the exits, and they know about each other's (un)truthfulness. The prisoner knows that one guard tells the truth and the other lies, but doesn't know which is which, and neither does he know which door leads to freedom. The king gives each prisoner one chance to ask one of the guards a single question which can be answered either "yes" or "no", in order to find the door to freedom. What should the prisoner ask?

For more of these puzzles, and an explanation as to why they might have a relevance to mathematical logic, look up the books of Raymond Smullyan.

Proof. The key is to ask one guard what the other guard will say about a question. Either guard will claim that the other one will lie. The honest guard because the other guard *will* lie, and lying guard because he lies about the honest guards truthfulness. Thus a question that the prisoner can ask is "Would the other guard say that the exit you are guarding leads to freedom?" Then if the answer is "yes", he choses the other door. It the answer is "no", he chose that door.

This is probably already a clear explanation, but just to eliminate all doubt we can divide the situation into four cases.

Case 1: The prisoner asks the truth telling guard who happens to be guarding the door to the torture chamber. Then the guard says, accurately, "yes". The lying guard will say that the exit leads to freedom. The prisonor chooses the other door.

Case 2:The prisoner asks the truth telling guard who happens to be guarding the door to freedom. Then the guard says that lying guard would say "no" because the lying guard would say that the door leads to the torture chamber. The prisoner chooses that door.

Case 3: The prisoner asks the lying guard who happens to be guarding the door to the torture chamber. Then the guard says, "yes", because the lying guard knows that the honest guard would say no.. The prisonor chooses the other door.

Case 4: The prisoner asks the lying guard who happens to be guarding the door that leads out to freedom. Then the guard says, "no" because the honest guard would say "Yes, this exit leads out.". The prisonor chooses that door. □