HOMEWORK 3: SOLUTIONS

PROBLEM 1

Let ϕ and ψ be formulas in propositional logic. Show that $\phi \vee \psi$ is equivalent to $\neg(\neg\phi \wedge \neg\psi)$.

Proof. Here a truth table suffices. Truth tables are pretty easy, and I'm not so good at tables in latex, so just ask me after class or in office hours if you want to see how this works. □

PROBLEM 2

In Problem 1, you showed that "or" can be expressed in terms of "and" and "not". Can you do the same with "implies"? That is, given a formula $\phi \to \psi$ in propositional logic, can you find an equivalent formula using ϕ, ψ, $\wedge$, and $\neg$? What about "if and only if"?

Proof. OK, the first step here is to do is another truth table: this time to show that that $(\phi \to \psi) \leftrightarrow \neg(p \wedge \neg q)$ is a tautology. Again, I'm not going do this in the solutions. Now since $p \leftrightarrow q$ is holds if and only if both $p \to q$ and $q \to p$ hold, we can use the first part of the problem to answer the second part as well: $p \leftrightarrow q$ is equivalent to $\neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p)$. □

PROBLEM 3

Prove that any formula in propositional logic is equivalent to a formula using only $\wedge$ and $\neg$.

Proof. Most of the work for this problem has been taken care of in Problems 1 and 2; all that is left is a simple inductive argument. The induction can be done on the length of the formula. In Problems 1 and 2 we have shown that any fomula of propositional logic of the form $p \vee q$, $p \to q$, and $p \leftrightarrow q$ is equivalent to a formula using only $\wedge$ and $\neg$. This forms the base case of the induction.

Now assume that the claim is true for formula of rank less than n. Suppose that ψ has rank n. Thenψ is of the form (1) $\varphi_1 \vee \varphi_2$, (2) $\varphi_1 \to \varphi_2$, (3) $\varphi_1 \leftrightarrow \varphi_2$, (4) $\varphi_1 \wedge \varphi_2$, or (5) $\neg\varphi_1$.

Assume we are in case (1). Note that φ_1 and φ_2 must be of lower rank than ψ. Thus, by the inductive hypothesis, there are $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$, using only $\wedge$ and $\neg$, that are equivalent to φ_1 and φ_2 respectively. Now, by applying Problem 1 again, we see that $\neg(\neg\widetilde{\varphi}_1 \wedge \neg\widetilde{\varphi}_2)$ is equivalent to ψ, and is the formula we seek.

Cases (2) and (3) are similar. Cases (4) and (5) are even easier, as $\varphi_1 \wedge \varphi_2$ can simply be replaced by $\widetilde{\varphi}_1 \wedge \widetilde{\varphi}_2$, and $\neg\varphi_1$ can simply be replaced with $\neg\widetilde{\varphi}_1$. (Here, I'm again using $\widetilde{\varphi}_i$ to indicate a formula equivalent to φ_1 using only $\wedge, \neg$.)

□

Problem 4

Fix an alphabet A, and suppose that A is countable. That is, there are a countable number of relation, function, and constant symbols. Prove that there are countably many formula in first order logic using the alphabet A..

Proof. The easiest thing to do is to show that a larger set is countable. Certainly if the set of all finite sequences of symbols from the alphabet A is countable, than the subset consisting of those that actually form well-formed formulas is also countable. Note that here we use the fact that I mentioned in the email to the class: that any formula is of finite length.

Let's fix a bijection, f, from A to $\mathbb{N}$. Let S_k be the set of all sequences from A of length k. The bijection f extends to a bijection between S_k and $\mathbb{N}^k$, which we have shown to be countable. Now if S is the set of all finite sequences from A, then $S = \bigcup S_k$, and we have shown that a countable union of countable sets is countable. □

Problem 5

Note that the word "buffalo" is both noun (big cow-like creatures that used wander the open plains in big herds) and a verb (either to intimidate with a display of power or to confuse). What's the longest English sentence you can make using only the word "buffalo"? Since this is a math course and shouldn't be testing your English language skills, I'll give a hint, but I'll wait until Monday to give people that want to think about it without a hint a chance to do so.

Proposition: Arbitrarily strings of the word "buffalo" are sentences in the English language.

Proof. To clarify the proof we will use "bison" as a synonym of "buffalo" as a noun, and "intimidate" as a synonym of "buffalo" as a verb.

Also we note that a noun followed by a verb can be used as an adjective, as in "Cats dogs chase run away." or "Students teachers like do their homework." In the first sentence "dogs chase" describe cats. In particular it limits the cats talked about to those chased by dogs. In the second sentence, "teachers like" performs the same role. We'll call such a noun-verb usage an *adjectival clause*.

Now we give a recursive construction of arbitrarily long sentences consisting only of the word "buffalo".

First note that "Buffalo buffalo buffalo." is a grammatical English sentence meaning "Bison intimidate bison."

Now assume that there is a sentence of length n consisting just of n uses of the word "buffalo". Choose a use of "buffalo" as a noun. We may assume that the first word in the sentence is a noun. Inserting "buffalo buffalo" after the first word in the sentence as an adjectival clause is a sentence of lenght $n + 2$ uses of the word "buffalo".

□