HOMEWORK 4

Problem 1

For each of the following formulas, write down (i) the list of subformulas, and (ii) the list of free variables.

(a). $\theta_1 := \forall x, y \exists z\, ((x < y) \to ((x < z) \wedge (z < y)))$

(b). $\theta_2 := \exists z(y + y < z)$

(b). $\theta_3 := \forall x((x < y) \vee (x = y)) \vee (x + x = x)$ (And, yes, that is how I want the parentheses.)

Problem 2

(a). Show that no matter what φ is, and no matter what $\mathfrak{M}$ is, $\mathfrak{M} \models \forall x\varphi \leftrightarrow \neg(\exists x \neg\varphi)$. A comment – two things are needed for this problem: first, unraveling what it means for a model to satisfy a sentence in this situation, and second, a fairly simple "English language" argument about the relation between "there exists" and "for all".

(b). Prove that any sentence in first order logic is equivalent to a sentence whose logical connectives are limited to $\exists, \neg$, and $\wedge$.

Problem 3

Let $\mathfrak{Z} := \{\mathbb{Z}, <\}$, where $\mathbb{Z}$ is the set of integers and $<$ is interpreted in the normal fashion. Let $\mathfrak{Q} := \{\mathbb{Q}, <\}$, where $\mathbb{Q}$ is the set of rational numbers and $<$ is interpreted in the normal fashion. Is there some sentence in first order logic such that $\mathfrak{Z} \models \varphi$ and $\mathfrak{Q} \not\models \varphi$. (By the way, the symbol "$\mathfrak{Z}$" is "Z" in the fraktur font, in case you were wondering.)

Problem 4

Let $\mathfrak{A} := \{(0,1), <\}$, where $(0,1)$ is the subset of real numbers greater than zero and less than one and $<$ is interpreted in the normal fashion. Let $\mathfrak{B} := \{[0,1] \cap \mathbb{Q}, <\}$, where $[0,1] \cap \mathbb{Q}$ is the set of rational numbers greater than or equal to zero and less than or equal to one, and $<$ is interpreted in the normal fashion. Is there some sentence in first order logic such that $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \not\models \varphi$?

Problem 5

In this problem we work in a language consisting of a single unary function S.
Let $\sigma_1 := \forall x \neg(S(x) = x)$.
Let $\sigma_2 := \forall x \neg(S(S(x)) = x)$, let $\sigma_3 := \forall x \neg(S(S(S(x))) = x)$, and so on, for each n
Let $\theta := \forall x \exists y(S(y) = x)$, and let $\psi := \forall x \forall y(S(x) = S(y)) \to (x = y))$.

Let $T := \{\psi, \theta, \sigma_1, \sigma_2, \ldots\}$.

Find a model for T. That is find a structure $\mathfrak{S}$ such that $\mathfrak{S} \models \phi$ for each sentence $\phi \in T$.

Problem 6

Here's another couple of logic puzzles leading up to the issues involved in Gödel's Incompleteness Theorem. The first is a warm up, the second starts considering the questions involved in Gödel's Incompleteness Theorem. Both take place on an island entirely populated by two types of people, knights and knaves. Knights only make true statements, knaves only make false statements. In every other respect, however, they are indistinguishable.

(a). A census taker is visiting the island, and has the task of counting how many knights, and how many knaves inhabit the island. The census taker approaches one house, and a rather timid man opens the door. All the census taker can get the man to say is "If I am a knight, then so is my wife". After some thought, (and perhaps a quick glance at his undergraduate notes on truth tables), the census taker realizes that he knows the types of both the man and his wife.

(b). A logician is visiting the island, and meets a native. The native says "You will never know that I am a knight". Does this lead to a paradox? (At the moment, this is not a very precise question. We will make it more precise in the future, but for now you could probably give arguments either direction. If your answer is "No, there is no paradox", why not? And are there additional assumptions you could make on the abilities of the logician that would make the answer, "Yes"? If you think it does lead to a paradox, try to be specific about what sort of reasoning the logician has to be capable of to reach a contradiction.