HOMEWORK 4 – SOLUTIONS

PROBLEM 1

For each of the following formulas, write down (i) the list of subformulas, and (ii) the list of free variables.

(a). $\theta_1 := \forall x, y \exists z\,((x < y) \to ((x < z) \wedge (z < y)))$

The subformulas are (i) $x < y$, (ii) $x < z$, (iii) $z < y$, (iv) $(x < z) \wedge (z < y)$, (v) $(x < y) \to ((x < z) \wedge (z < y))$, (vi) $\exists z\,((x < y) \to ((x < z) \wedge (z < y)))$, (vii) $\forall y(\exists z\,((x < y) \to ((x < z) \wedge (z < y))))$, and (viii) $\forall x, y \exists z\,((x < y) \to ((x < z) \wedge (z < y)))$.

There are no free variables.

(b). $\theta_2 := \exists z(y + y < z)$

The subformulas are (i) $y + y < z$, and (ii) $\exists z(y + y < z)$.

The free variable is y.

(c). $\theta_3 := \forall x((x < y) \vee (x = y)) \vee (x + x = x)$ (And, yes, that is how I want the parentheses.)

The subformulas are (i) $x < y$, (ii) $x = y$, (iii) $x + x = x$, (iv) $(x < y) \vee (x = y)$, (v) $\forall x((x < y) \vee (x = y))$, and (vi) $\forall x((x < y) \vee (x = y)) \vee (x + x = x)$.

Both x and y are free. It's probably clear why y is free. But why is x free?

Let φ be $\forall x((x < y) \vee (x = y))$. Note that x is not free in φ, but it is free in $x + x = x$. Since, by definition, the free variables of θ_3 are the free variables of φ union the free variables of $x + x = x$, we see that x is a free variable of θ_3.

Intuitively, it's because the $\forall x$ quantifies only over φ, leaving the x in $x + x = x$ free.

PROBLEM 2

(a). Show that no matter what φ is, and no matter what $\mathfrak{M}$ is, $\mathfrak{M} \models \forall x \varphi \leftrightarrow \neg(\exists x \neg\varphi)$. A comment – two things are needed for this problem: first, unraveling what it means for a model to satisfy a sentence in this situation, and second, a fairly simple "English language" argument about the relation between "there exists" and "for all".

Proof. First we note that $\mathfrak{M} \models \theta_1 \leftrightarrow \theta_2$ if and only if whenever $I = (\mathfrak{M}, \beta)$ is an interpretation, $I \models \theta_1$ precisely when $I \models \theta_2$. Thus we need to show that given any interpretation, I, we have $I \models \forall x \varphi$ precisely when $I \models \neg(\exists x \neg\varphi)$

Going back to the definition, we see that $I \models \neg(\exists x \neg\varphi)$ happens iff it is not the case that $I \models \exists x \neg\varphi$. And unwinding the definition one more step, we see that this occurs iff it is not the case that there is some $a \in M$ such that $I\frac{a}{x} \models \neg\varphi$.

But this occurs precisely if for all $a \in M$, we do not have that $I\frac{a}{x} \models \neg\varphi$. Going to the definition once again, this occurs precisely when for all $a \in M$, we have that $I\frac{a}{x} \models \varphi$. One final application of the definition gives us that this occurs precisely when $I \models \forall x \varphi$.

What have we done in this proof? We have taken our definition, which deals with the symbols "$\forall$" and "$\exists$" and showed that something that we know holds for statements "for all x ..." and "there exists x . . ." makes sense for the symbols as well. (Which more or less means our definition was a good one.) $\square$

(b). Prove that any sentence in first order logic is equivalent to a sentence whose logical connectives are limited to $\exists, \neg$, and $\wedge$.

Proof. This is another argument by induction on the rank of formulas.

First we deal we the atomic formulas: that is with formulas of the form $R(t_1, \dots, t_n)$ or $t_1 = t_2$. But here there is nothing to prove.

Now assume that the claim is true for formula with rank less than n, and let φ be a formula of rank n. By previous homeworks, we may assume that φ is in one of four forms: (i) $\theta_1 \wedge \theta_2$, (ii) $\neg\theta$, (iii) $\exists x\theta$, or (iv) $\forall x\theta$; and that in each case, θ (being of rank less than n) only uses the logical connectives $\exists, \neg$, and $\wedge$.

In the first three cases, there is again nothing to prove. Thus we assume that φ is $\forall x\theta$. By part (a), $\forall x\theta$ is equivalent to $\neg\exists x\neg\theta$, and we are done. $\square$

PROBLEM 3

Let $\mathfrak{Z} := \{\mathbb{Z}, <\}$, where $\mathbb{Z}$ is the set of integers and $<$ is interpreted in the normal fashion. Let $\mathfrak{Q} := \{\mathbb{Q}, <\}$, where $\mathbb{Q}$ is the set of rational numbers and $<$ is interpreted in the normal fashion. Is there some sentence in first order logic such that $\mathfrak{Z} \models \varphi$ and $\mathfrak{Q} \not\models \varphi$. (By the way, the symbol "$\mathfrak{Z}$" is "Z" in the fraktur font, in case you were wondering.)

The answer is "yes".

Proof. There's not much to prove, once you see the formula. $\mathfrak{Q} \models \forall x, y(x < y) \to (\exists z(x < z) \wedge (z < y))$. This is clear enough that you don't really need prove that this is the case. But if you wanted to you could take any x and y in $\mathbb{Q}$ and note that $z := \frac{x+y}{2}$ works. On the other hand it is clear that $\mathfrak{Z}$ does not satisfy this sentence. Take x and y equal to 1 and 2 respectively. Then it is clear that there is no $z \in \mathbb{Z}$ such that z is between x and y. $\square$

PROBLEM 4

Let $\mathfrak{A} := \{(0, 1), <\}$, where $(0, 1)$ is the subset of real numbers greater than zero and less than one and $<$ is interpreted in the normal fashion. Let $\mathfrak{B} := \{[0, 1] \cap \mathbb{Q}, <\}$, where $[0, 1] \cap \mathbb{Q}$ is the set of rational numbers greater than or equal to zero and less than or equal to one, and $<$ is interpreted in the normal fashion. Is there some sentence in first order logic such that $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \not\models \varphi$?

Again the answer is "yes".

Proof. And again, there's not much to prove once you find the formula. $\mathfrak{A} \models \forall x \exists z x < z$, since there is no greatest element in $(0, 1)$. On the other hand, letting $x := 1$ shows that $\mathfrak{B} \not\models \forall x \exists z x < z$. $\square$

PROBLEM 5

In this problem we work in a language consisting of a single unary function S.
Let $\sigma_1 := \forall x \neg(S(x) = x)$.
Let $\sigma_2 := \forall x \neg(S(S(x)) = x)$, let $\sigma_3 := \forall x \neg(S(S(S(x))) = x)$, and so on, for each n
Let $\theta := \forall x \exists y (S(y) = x)$, and let $\psi := \forall x \forall y (S(x) = S(y)) \rightarrow (x = y))$.

Let $T := \{\psi, \theta, \sigma_1, \sigma_2, \ldots\}$.

Find a model for T. That is find a structure $\mathfrak{S}$ such that $\mathfrak{S} \models \phi$ for each sentence $\phi \in T$. The easiest way to present a model for this theory would be to say let $\mathfrak{S} := (\mathbb{Z}, S)$ where $S(x) = x + 1$. Checking that $\mathfrak{S} \models T$ is also not difficult: in this model σ_n just says that $x + n$ is never equal to x. One should note that θ just says that S is surjective, and ψ just says that S is injective. It's pretty clear that the map $x \mapsto x + 1$ is a bijection, so that confirms that $\mathfrak{S}$ satisfies all of T.

Note that we didn't have to use the integers as the universe of our model. We could just as easily let the universe be any countable set A. Write A as $\{a_i : i \in \mathbb{Z}$ and let S be the map that sends a_i to a_{i+1}. We could even let A be uncountable, but that is not quite as clear.

1. PROBLEM 6

Here's another couple of logic puzzles leading up to the issues involved in Gödel's Incompleteness Theorem. The first is a warm up, the second starts considering the questions involved in Gödel's Incompleteness Theorem. Both take place on an island entirely populated by two types of people, knights and knaves. Knights only make true statements, knaves only make false statements. In every other respect, however, they are indistinguishable.

(a). A census taker is visiting the island, and has the task of counting how many knights, and how many knaves inhabit the island. The census taker approaches one house, and a rather timid man opens the door. All the census taker can get the man to say is "If I am a knight, then so is my wife". After some thought, (and perhaps a quick glance at his undergraduate notes on truth tables), the census taker realizes that he knows the types of both the man and his wife.

The census taker can reason as follows: Suppose that the the man is a knave. Then what he said is false. The negation of "p implies q" is "p and not q" so I can conclude that the man is a knight, and that his wife is a knave. But then in particular, the man is not a knave, so my supposition was incorrect. Thus the man is a knight. And therefore, so is his wife.

(b). A logician is visiting the island, and meets a native. The native says "You will never know that I am a knight". Does this lead to a paradox? (At the moment, this is not a very precise question. We will make it more precise in the future, but for now you could probably give arguments either direction. If your answer is "No, there is no paradox", why not? And are there additional assumptions you could make on the abilities of the logician that would make the answer, "Yes"? If you think it does lead to a paradox, try to be specific about what sort of reasoning the logician has to be capable of to reach a contradiction.

The logician reasons as follows: Suppose he is a knave. Then his statement is false, which means that at some point I *will* know that he is knight. But I can't *know* that he is a knight unless he really is one. But then he wouldn't be a knave. Thus I may conclude that he is a knight. On the other hand, he has said that I will never know that he is a knight, but I have just concluded that he is a knight, thus his statement was false. Therefore he must be a knave!

What assumptions have a made about the logician in order to reach the contradiction? On the one hand I have made several pragmatic assumptions. For instance there is no contradiction if around the time the logician is saying to himself "Suppose he is a knave . . ." a bolt of lightning slays the logician. More interestingly from our perspective, we are making assumptions on the logicians reasoning capability. If the logician were not a terribly bright fellow, he might have pondered the statement for a while, not come up with anything, and gone on his way.

We will call an individual a *reasoner of type 1* if he or she

(1) believes all tautologies of propositional logic, and

(2) For any propostions p and q, if he believes p and he believes $pimpliesq$ then he believes q.

Thus we have established the following:

Given a knight-knave island, and given a reasoner of type 1 who knows the rules of the island and who hears what is said to him, it is impossible for any native to say to him "You will never correctly believe that I am a knight."

One brief remark: As I described it, the logician has a certain degree of self-awareness: after concluding that the native was a knight, the logician knew that he knew that the native was a knight, and thus realized that there was a contradiction. But even had the logician not known that there was a contradiction, there still would have been one, so we did not need to add this requirement to the conditions that makes someone a type 1 reasoner.