

HOMEWORK 5

PROBLEM 1

In this problem we return the language and theory described in Problem 5 of last week's homework. To remind you, L is a language consisting of a single unary function S , and T consists of the following sentences:

$$\sigma_1 := \forall x \neg(S(x) = x).$$

$$\sigma_2 := \forall x \neg(S(S(x)) = x), \text{ let } \sigma_3 := \forall x \neg(S(S(S(x))) = x), \text{ and so on, for each } n$$

$$\theta := \forall x \exists y(S(y) = x), \text{ and let } \psi := \forall x \forall y(S(x) = S(y)) \rightarrow (x = y).$$

Last week you showed that T was consistent by finding a model for T . This week, show that no finite subset of T implies all of T .

To do this, we have to show that any given finite subset of T has a model that doesn't satisfy all of T . Let $T_n := \{\sigma_1, \dots, \sigma_n, \theta, \psi\}$. Note that any finite subset of T is contained in T_n for some n . Thus it suffices to find a model \mathfrak{S}_n such that $\mathfrak{S}_n \models T_n$, but $\mathfrak{S}_n \not\models T_{n+1}$.

Let $\mathfrak{S}_n = (\{1, \dots, n+1\}, S)$ where S is now defined as follows: for $1 \leq x \leq n$ let $S(x) := x+1$, and for $x = n+1$ let $S(x) := 1$.

It is clear that S is still a bijection, so $\mathfrak{S}_n \models \theta \wedge \psi$. Also repeating S n -times never gets you back to where you started, so $\mathfrak{S}_n \models \sigma_1 \wedge \dots \wedge \sigma_n$. However, $\mathfrak{S}_n \not\models \sigma_{n+1}$ because applying S $n+1$ times to any element x gives you that element back again.

PROBLEM 2

Find a finite set of sentences of first order logic (in any language you like), such that the any model of the finite set of sentences is infinite. I'll give a hint on Monday (although you may find that it is not too hard.).

There are many ways to do this problem. For instance you could use the axioms of linear orders given in class, and add the sentence $\forall x, y(x < y) \rightarrow (\exists z(x < z) \wedge (z < y))$ from Problem 3. Now, any model that satisfies these four sentences, is what is called a *dense linear order*, which just means that it is a linear order and that between any two points there is a third point. One can then show that dense linear orders must be infinite.

Perhaps even easier is to just take the two sentences ψ and $\neg\theta$, where ψ and θ are as in the previous problem. I'll do this example in detail.

We let L be the language consisting of a single unary function S . Let $\mathfrak{A} := (A, S)$, and suppose that $\mathfrak{A} \models \psi \wedge \neg\theta$. Thus, S is an injective function defined on the universe of A , and S is not surjective. Therefore S gives a bijection between A and $\text{Im}S$, and thus $|A| = |\text{Im}S|$. But the image of S is a proper subset of A (since S is not surjective). Since A is the same size as a proper subset of itself, A is infinite.