HOMEWORK 6 – SOLUTIONS

Problem 1

Define the equivalence relation $\sim$ on $\mathbb{Z}$ as follows: Let $a \sim b$ iff $a - b$ is divisible by 7.

(1). Show that this is indeed an equivalence relation.

Proof. One needs to prove three things (1) for all x, $x \sim x$, (2) for all x and y, $x \sim y$ iff $y \sim x$, and (3) for all x, y, z, $x \sim y$ and $y \sim z$, implies $x \sim z$.

We'll start with (1). By definition, $x \sim x$ means that $x - x$ is divisible by seven. But clearly $x - x = 0$, and 0 is divisible by seven.

Note that $x \sim y$ iff $x - y$ is divisible by seven iff $-(x - y)$ is divisible by seven iff $y - x$ is divisible by seven iff $y \sim x$. Thus we have shown (2).

To prove (3), take $x \sim y$ and $y \sim z$. To be more specific, let $x - y = 7m$, and $y - z = 7n$. Note that $x - z = (x - y) + (y - z) = 7m + 7n = 7(m + n)$, so $x - z$ is divisible by seven, and hence we have proven (3). □

(2). Define $+$ on $\mathbb{Z}/\sim$ as follows: Let x, y be two elements of $\mathbb{Z}/\sim$ (i.e. two equivalence classes of $\sim$ in $\mathbb{Z}$). Pick some $n \in x$ and $m \in y$. Define $x + y$ to be the equivalence class to which $m + n$ belongs. Prove that this function is well-defined. Give a similar definition for multiplication, and prove that this too is well-defined. (To all those of you who have done this before, I apologize if this is boring.)

Proof. First to show that $+$ is well-defined on $\mathbb{Z}/\sim$. Take n_1, n_2 in x and m_1, m_2 in y. We must show that $(n_1 + m_1) \sim (n_2 + m_2)$. Since $n_1 \sim n_2$ we know that $n_1 - n_2 = 7a$ for some $a \in \mathbb{Z}$. Thus we can write n_1 as $n_2 + 7a$. Likewise $m_1 = m_2 + 7b$ for some $b \in \mathbb{Z}$. By definition, what we need to show is that $n_1 + m_1 - (n_2 + m_2)$ is divisible by 7. But this is just the claim that $n_2 + 7a + m_2 + 7b - (n_2 + m_2) = 7(a + b)$ is divisible by 7, which is obvious.

Now to define multiplication on $\mathbb{Z}/\sim$. This is done in the same fashion as addition. Take $x, y \in \mathbb{Z}$. Take $n \in x$, take $m \in y$, and define $x \cdot y$ to be the equivalence class of $n \cdot m$. For this definition to make sense, it can't depend on the choice of m and n. So take n_1, n_2 in x, and $m_, m_2$ in y. We have to show that $n_1 m_1 - n_2 m_2$ is divisible by 7. As before, write n_1 as $n_2 + 7a$, and m_1 as $m_2 + 7b$. Now we need to show that $(n_2 + 7a)(m_2 + 7b) - n_2 m_2$ is divisible by 7. But $(n_2 + 7a)(m_2 + 7b) - n_2 m_2 = n_2 m_2 + 7bn_2 + 7am_2 + 49ab - n_2 m_2 = 7(am_2 + bn_2 + 7ab)$ and we are done.

□

Problem 2

Fix a language L, and let f be a unary function in the language. Show that if a finite collection of formulas, Γ proves $t_1 = t_2$ then Γ proves that $f(t_1) = f(t_2)$.

Proof. $\Gamma\ t_1 = t_2$ Premise
$f(t_1) = f(t_1)$ Equality
$\Gamma\ f(t_1) = f(t_2)$ Ant.
$\Gamma\ t_1 = t_2\ f(t_1) = f(t_2)$ Sub. applied to $f(t_1) = x|\frac{t_2}{x}$
$\Gamma\ f(t_1) = f(t_2)$ Chain Rule applied to the two previous lines. □

Problem 3

Suppose that T is a consistent set of sentences, and φ is any formula, then either $T \cup \{\varphi\}$ is consistent, or $T \cup \{\neg\varphi\}$ is consistent (or both, of course).

Proof. Suppose, for a contradiction, that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are inconsistent. That is there is some θ_1 such that $T \cup \{\varphi\}$ proves both θ_1 and $\neg\theta_1$, and that there is some θ_2 such that $T \cup \{\neg\varphi\}$ proves both θ_2 and $\neg\theta_2$.

The first thing to do is to show that we may assume that $\theta_1 = \theta_2$. Thus we have the following lemma: (which, by the way is in the book, and so you could certainly use it without proving it, but I'm including it here for those without the book)

Lemma 0.1. *If T is inconsistent iff for all ψ, T proves ψ.*

Proof. By definition T is inconsistent iff there exists φ such that T proves both φ and $\neg\varphi$. Clearly, if T proves every formula, this happens. Thus right implies left is clear.

To prove that left implies right, we assume that T is inconsistent, i.e. that T proves both θ and $\neg\theta$, and pick arbitrary ψ and try to show that T proves ψ.

Since T proves θ, there is some finite $\Gamma_1 \subseteq T$ that proves θ, and likewise there is some finite Γ_2 that proves $\neg\theta$. Let $\Gamma := \Gamma_1 \cup \Gamma_2$. Then Γ proves both θ and $\neg\theta$. (This latter assertion follows from the sequent calculus rule Antecedent). Now we have:
$\Gamma\ \theta$ Premise
$\Gamma\ \neg\theta$ Premise
$\Gamma\ \neg\varphi\ \theta$ (Ant)
$\Gamma\ \neg\varphi\ \neg\theta$ (Ant)
$\Gamma\ \varphi$ (Ctr)
Come to think of it, I proved this in class, and called this fact (Ctr'). □

Now that we have proven the lemma, we may return to our proof. We have assume for a contradiction that both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are inconsistent. We reach a contradiction by proving that T is inconsistent. We have the following: $T \cup \{\varphi\}$ proves θ for all θ and $T \cup \{\neg\varphi\}$ also proves θ for all θ. In fact, we can find some finite $\Gamma \subseteq T$ such that we have for any θ:
(1) $\Gamma\ \varphi\ \theta$
(2) $\Gamma\ \neg\varphi\ \theta$
Now we immediately have:
$\Gamma\ \theta$ by (PC).
Thus Γ is inconsistent, and thus T, which contains Γ is inconsistent, and we have our contradiction.

□