HOMEWORK 7 – SOLUTIONS

Problem 1

Pick a language L, and pick an inconsistent Φ consisting of formulas from the language L. Construct $\mathfrak{T}$ as in class. Does $\mathfrak{T}$ depend on T, the set of terms? Does $\mathfrak{T}$ model Φ?

I'm just going to let L be the language of equality to make life easier on myself. That is, L is the language with no relation, function, or constant symbols. Let $T := \{\exists x(x \neq x)\}$. (This is inconsistent since it, like all other theories, proves $\neg\forall x(x \neq x)$).

There are no function symbols or constants, so the only terms are the variables. Thus the universe of $\mathfrak{T}$ is just the set of variables modulo $\sim$. Thus we have to figure out when Φ proves $v_1 = v_2$. But, as we have previously shown, an inconsistent theory proves everything. Thus every term is equivalent to every other one. Thus the universe of $\mathfrak{T}$ has a single element.

Clearly, all that we used was the fact that Φ is inconsistent. Thus no matter which Φ we choose and what T, the set of terms is, as long as Φ is inconsistent, $\mathfrak{T}$ will have the same universe. Furthermore, each relation in the language will hold of the single element in the universe of the model, and each function symbol will be the identity function.

And, of course, $\mathfrak{T}$ doesn't satisfy Φ as it is impossible to satisfy a sentence like $\forall x(x \neq x)$.

Problem 2

Pick a language L and for each term in the language let φ_t be the formula $v_0 = t$. Let θ be the formula $\exists v_0 \exists v_1(\neg(v_0 = v_1))$. Let Φ be the collection of all φ_t together with θ. Show that Φ is consistent, but there is no consistent set of L formulas containing Φ which contains witnesses.

Proof. First to show that Φ is consistent. I only asked you to do this for one given language, so you can make your life easier by picking a language without many functions, relations, or constants, but here I will give a proof for arbitrary L. Let L consist of some collection of relations $\{R_i\}$, functions $\{f_j\}$, and constants $\{c_k\}$. Let $\mathfrak{M}$ have a universe consisting of two elements, a and b. Let each constant, c_k designate a, let each f_j be the function that always outputs a, and it will not matter what the relations are. Now consider the interpretation $I := (\mathfrak{M}, \beta)$, where β maps each variable to a.

Now let's consider whether $I \models T$, and thus shows that T is consistent. By definition, I models $v_0 = t$ iff $\beta(v_0) = \beta(t)$. We know that $\beta(v_0)$ is a. Thus we must show that $\beta(t)$ is also a for each term t. Suppose that t is a constant. Then $\beta(t) = c$, but since each constant symbol designates a, this case works.

Now suppose that t is a variable, v_i. Then $\beta(v_i) = \beta(v_i) = a$. Now suppose that $t = f_k(t_1, \ldots, t_n)$. But then $t = a$ since f_k applied to anything is a. Thus, we have shown that $I \models \varphi_t$ for each t.

Now we have to show that $I \models \theta$. By definition, $I \models \exists v_0, v_1(\neg(v_0 = v_1))$ iff there is some $m \in \mathfrak{M}$ such that $(\mathfrak{M}, \beta\frac{m}{v_0}) \models \exists v_0, (\neg(v_0 = v_1))$. Applying the definition one more time, we see that $(\mathfrak{M}, \beta\frac{m}{v_0}) \models \exists v_0, (\neg(v_0 = v_1))$ iff there is m' in $\mathfrak{M}$ such that $(\mathfrak{M}, \beta\frac{mm'}{v_0 v_1}) \models \neg(v_0 = v_1)$. If one lets $m := a$ and $m' = b$, it is easy to see that this is true. Thus $I \models \theta$ and we have shown that T is consistent.

We must now show that there is no way to extend T to a larger collection of sentences that contains witnesses. If $T' \supset T$ contains witnesses, then there must be some t_0 such that T' proves $\exists v_1(\neg(t_0 = v_1))$. And thus T' must also prove $\neg(t_0 = t_1)$ for some term, t_1. But note that T contains $t_0 = v_0$, and $t_1 = v_0$. We have previously shown that the sequent calculus proves that $=$ is an equivalence relation, and thus we may conclude that T proves $t_0 = t_1$. And thus T' must be inconsistent.

□

PROBLEM 3

Show that if T has arbitrarily large countable models, then T has an infinite model. Is there a theory T true of all finite groups, but not true of any infinite groups?

Proof. This problem was meant to read: recall that if T has arbitrarily large finite models, then T has an infinite model. What was written is true, but in trivial fashion. And not very useful as a hint.

Also I should point out that, as should hopefully be clear, that here I have once again forgotten, and written T to mean a collection of sentences, not a collection of terms.

On the other hand, if you could make out what the hint meant, the problem is very easy. Even if you didn't figure out what I meant by the hint, the problem is not so hard. It just points out an application of what we have proven.

The problem requires you to prove that if T is true of all finite groups then T is true of some infinite group. Using the correct hint, this consists of observing that being a group is a first order property, so any infinite model of T is the desired infinite group.

I will also give a proof that does not use what Salih did when he taught the class for me. This proof will use the compactness theorem, and we will begin by expanding the language of groups (that is $L = \{\circ, e\}$) to the language L' consisting of the language of groups together with a countably infinite collection of new constants $\{c_n : n \in \mathbb{N}\}$. We extend T to T' by adding the sentences saying that each c_n is distinct. That is we add the statements $c_i \neq c_j$ for each $i \neq j$. We also add to T' any of the group axioms not in T.

Now we must show that T' has a model. Take some finite subset of T'. This finite subset will mention some finite number, say N, of the new constants. Now take a finite group G_N of size greater than N, and let the constants mentioned in our finite subset of T' name distinct elements of G_N. Clearly G_N satisfies our finite subset of T'. Thus the compactness theorem tells us that there is some model G' of T'. Clearly this model is an infinite group. Now let G be identical to G' except that G will be in the language of groups, i.e. G will not have names for the constants c_n. Clearly $G \models T$ and we are done.

Note that the fact that we were dealing in groups in particular never came into the proof. All that was required was the T had arbitrarily large finite models. $\square$

Problem 4

Let $L := \{+, \cdot, 0, 1, <\}$.

(a). Show that each rational number is definable in $(\mathbb{R}, +, \cdot, 0, 1, <)$. Note that I am not asking for you to show that the set of all rational numbers is a definable set, but rather each individual rational number is a definable element.

Proof. First note that any natural number is definable as $x = 1 + \ldots + 1$ for an appropriate number of 1's. Next note that any negative integer is definable as $x + n = 0$, where n is an appropriate natural number. And finally note that any rational number is definable as $mx = n$, where m and n are appropriate integers. $\square$

(b). Show that there is a model $\mathfrak{R}$ such that $\mathfrak{R}$ and $(\mathbb{R}, +, \cdot, 0, 1, <)$ satisfy precisely the same set of formulas, and such that there is an element of $\mathfrak{R}$ greater than zero and less than every positive rational number. (Such an element is called an $infinitessimal$ element.")

Proof. Again, a proof by the compactness theorem. Expand the language by adding a single new constant, c. For each positive rational number, q, add the sentence $c < q$ to the set of sentences true of $\mathbb{R}$ and call this theory T'. Take a finite subset of T'. Let q_0 be the smallest rational number mentioned in this finite subset of T'. Let $\mathbb{R}_{q_0}$ be the real numbers with c naming some number less than q_0. Clearly $\mathbb{R}$ satisfies our finite subset of T'. Applying the compactness theorem as in the previous two problems gives us our proof. $\square$