HOMEWORK 8

Problem 1

Let $L := \{R\}$ with R a unary relation, and let $\Phi := \{R(x) \vee R(y)\}$ where x and y are distinct variables.

(a). Show that it is not the case that Φ proves $R(x)$, and that it is not the case that Φ proves $R(y)$ (i.e. Φ is not negation complete).

(b). Now construct the model $\mathfrak{T}$ as in class. That is, the universe of $\mathfrak{T}$ is the set of terms mod $\sim$, where $t_1 \sim t_2$ iff Φ proves $t_1 = t_2$, and one defines $R(t)$ to hold in $\mathfrak{T}$ iff Φ proves $R(t)$. Show that $\mathfrak{T}$ does not model Φ.

Problem 2

Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are models in the same language. Suppose that $Th(\mathfrak{M}) \subseteq Th(\mathfrak{N})$. Show that $\mathfrak{M}$ is elementarily equivalent to $\mathfrak{N}$.

Problem 3

(a). Let $\mathfrak{R}$ be a model elementarily equivalent to $(\mathbb{R}, +, \cdot, 0, 1)$. Show that there is an injection, f, of the rational numbers into $\mathfrak{R}$ such that $f(0) = 0$, $f(1) = 1$, $f(x + y) = f(x) + f(y)$, and $f(xy) = f(x)f(y)$. Furthermore show that this map is unique. (This isn't meant to be a hard problem. It's only meant to let me talk about the rational numbers as a subset of $\mathfrak{R}$ in part (b))

(b). Show that there is a model elementarily equivalent to $(\mathbb{R}, +, \cdot, 0, 1)$ with an element greater than all of the rational numbers. Is it possible to find such a model that has no infinitessimal elements?

Problem 4

In the past exercises, definable elements of a model have been precisely those fixed by all of the automorphisms. While there are circumstances where this is true, it is not always the case. This exercise gives a counterexample.

(a). How many elements of $(\mathbb{R}, \cdot, +, <, 0, 1)$ are definable? What is the cardinality of $\mathbb{R}$? What is therefore the cardinality of the set of elements of $(\mathbb{R}, \cdot, +, 0, 1)$ which are not definable?

(b). Let f be an automorphism of $(\mathbb{R}, \cdot, +, <, 0, 1)$. Prove that the inverse image of any interval is itself an interval (i.e. the automorphism is continuous). Note that the f fixes all rational numbers (why?), and hence maps each interval with rational endpoints to itself. Finally use a proof by contradiction to show that f must map each real number to itself.

Problem 5

Let A be an alphabet. Let A^* be the set of finite strings of symbols from A. Let W and W' be decidable subsets of A^*. Show that $W \cup W'$, $W \cap W'$, and $A^* \setminus W$ are all decidable.

Problem 6

Describe the decision procedure for the set of formulas of a language L, where L has a finite number of relation, function and constant symbols.