HOMEWORK 8

Problem 1

Let $L := \{R\}$ with R a unary relation, and let $\Phi := \{R(x) \vee R(y)\}$ where x and y are distinct variables.

(a). Show that it is not the case that Φ proves $R(x)$, and that it is not the case that Φ proves $R(y)$ (i.e. Φ is not negation complete).

Proof. To show that T does not prove $R(x)$, we find an interpretation $I := (\mathfrak{M}, \beta)$ of T that satisfies T, but such that $I \models \neg R(y)$. This time let's use an example of a model that isn't a collection of numbers of some sort. Let $\mathfrak{M} := (M, R)$ where M is the set of universities in the United States, and let $R(m)$ mean that m is in the Big Ten. Let β map x to Notre Dame and y to Illinois. Then clearly, $I \models \neg R(x)$, but since Illinois is in the Big Ten, $I \models T$.

To show that T does not prove $R(y)$ just pick β which switches what x and y are mapped too.

I think it's clear that T does not prove either $\neg R(x)$ or $\neg R(y)$, but that can proven in a similar fashion if desired.

Thus T is not complete (for example T does not prove either $R(x)$ or $\neg R(x)$). $\square$

(b). Now construct the model $\mathfrak{T}$ as in class. That is, the universe of $\mathfrak{T}$ is the set of terms mod $\sim$, where $t_1 \sim t_2$ iff Φ proves $t_1 = t_2$, and one defines $R(t)$ to hold in $\mathfrak{T}$ iff Φ proves $R(t)$. Show that $\mathfrak{T}$ does not model Φ.

Again the point of this problem is to illustrate why we made the choices we did when proving the Completeness Theorem. One of the things that we did was to take the theory with which we started and enlarge it to a complete theory. This problem shows that if we had not done this, our proof would have failed.

As in part (a), the only terms in the language are the variables. It is not difficult to see that T does not prove that $x = y$ for any two distinct variables x and y. Thus the universe of $\mathfrak{T}$ is just the set of variables.

The argument in part (a) above, shows that T does not prove $R(z)$ for any variable z. Thus R holds of nothing in the universe of $\mathfrak{T}$. But then $\mathfrak{T}$ cannot model T. In fact, no matter what interpretation $I := (\mathfrak{T}, \beta)$ is chosen, it is not the case that $I \models T$, since β cannot map either x or y to an element of which R holds.

Problem 2

Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are models in the same language. Suppose that $Th(\mathfrak{M}) \subseteq Th(\mathfrak{N})$. Show that $\mathfrak{M}$ is elementarily equivalent to $\mathfrak{N}$.

Proof. The key to this problem lies in the fact that the theory of a model is both complete and consistent.

We have that $Th(\mathfrak{M}) \subseteq Th(\mathfrak{N})$. We need to show that $Th(\mathfrak{N}) \subseteq Th(\mathfrak{M})$. Suppose that it isn't. Take some θ such that θ is in $Th(\mathfrak{N})$ but not in $Th(\mathfrak{M})$.

Since θ is not true about $\mathfrak{M}$, it must be the case that $\neg\theta$ is true. But then $\neg\theta$ is in $Th(\mathfrak{M})$ and thus in $Th(\mathfrak{N})$ as well. But this contradicts the fact that $Th(\mathfrak{N})$ is consistent. □

PROBLEM 3

(a). Let $\mathfrak{R}$ be a model elementarily equivalent to $(\mathbb{R}, +, \cdot, 0, 1)$. Show that there is an injection, f, of the rational numbers into $\mathfrak{R}$ such that $f(0) = 0$, $f(1) = 1$, $f(x + y) = f(x) + f(y)$, and $f(xy) = f(x)f(y)$. Furthermore show that this map is unique. (This isn't meant to be a hard problem. It's only meant to let me talk about the rational numbers as a subset of $\mathfrak{R}$ in part (b))

Proof. First let me say a little bit about why I assigned this problem. In several examples from class and also in the homework, I assumed that not only are the rational numbers a subset of $\mathbb{R}$, but they are also a subset of any model elementarily equivalent to $\mathbb{R}$. In a certain technical sense, this is false. But it is true in the sense that you are proving in this problem. That is, there is a unique copy of the rational numbers in any field elementarily equivalent to $\mathbb{R}$.

First of all, recall that each rational number is definable in $\mathbb{R}$. Let's say that the rational number q is defined by the formula $\varphi_q(x)$. One question that one might ask is whether $\varphi_q(x)$ also defines an element of $\mathfrak{R}$. After all, perhaps the subset of $\mathfrak{R}$ defined by $\varphi_q(x)$ contains more than one element. But this is easily seen not to be true, as both $mathbbR$ and $\mathfrak{R}$ satisfy the sentence that says "there exists precisely one x such that $\varphi_q(x)$.

Thus $varphi_q(x)$ defines an element of $\mathfrak{R}$ and it makes sense to map q to the element of $\mathfrak{R}$ defined by $varphi_q(x)$. It is easy to see that this is an injection of the form asked for in the problem.

Now for uniqueness: this follows from the fact that once you know that $f(0) = 0$ and $f(1) = 1$, there is no choice as to where any other $q \in \mathbb{Q}$ gets mapped. For instance, for $n \in \mathbb{N}$, since $n = 1 + \ldots + 1$, and since we have required that f preserve addition, one must map n to $f(1) + \ldots + f(1)$ which equals $1 + \ldots + 1$. Which, as mentioned above, technically one should think of as "the unique x satisfying the formule $varphi_n(x)$, but as we are in the process of proving, one can also just think of as "n".

If $z \in \mathbb{Z}$ then $z = -1 \cdot n$ for some $n \in \mathbb{N}$, and thus f(z) must equal $f(-1) \cdot f(n) = -f(n)$. Likewise it is easy to show that $\frac{m}{n}$ must be mapped to $fracf(m)f(n)$. □

(b). Show that there is a model elementarily equivalent to $(\mathbb{R}, +, \cdot, 0, 1)$ with an element greater than all of the rational numbers. Is it possible to find such a model that has no infinitessimal elements?

Proof. This proof is a compactness argument very similar to the one in HW 6 in which one finds a model elementarily equivalent to $\mathbb{R}$ that has an infinitessimal element. I'll just sketch the proof. Expand the language to L' by adding a new constant c. Expand $Th(\mathbb{R})$ by saying that $c > n$ for each natural number n. Any finite subset of these statements has a model: just let c be a real number bigger than the biggest n mentioned in the finite subset. Thus by the compactness theorem, there is a model with an element bigger than each natural number. (Note that I'm using Problem 2a to assume that n is in my new model.)

The interesting part of this problem is whether or not you can have a model with an infinitely large element but no infinitessimal element. One might very well try to use the compactness theorem to prove the existence of such a model. But here one will run into the difficultly that the compactness theorem is much better at building models where something exists rather than ones where something does not exist. Often it can be quite difficult to tell whether you can find a model where you insist that one thing happens, but require something else does not happen.

This, however, is not one of those times where it is difficult to tell what happens. The additive inverse of an infinitessimal element is bigger than any natural number, and vice versa. So if you have one you have the other.

□

Problem 4

In the past exercises, definable elements of a model have been precisely those fixed by all of the automorphisms. While there are circumstances where this is true, it is not always the case. This exercise gives a counterexample.

(a). How many elements of $(\mathbb{R}, \cdot, +, <, 0, 1)$ are definable? What is the cardinality of $\mathbb{R}$? What is therefore the cardinality of the set of elements of $(\mathbb{R}, \cdot, +, 0, 1)$ which are not definable?

Proof. We know that rationals are definable, so there are at least countably infinite (that is $\aleph_0$ many) definable elements. But we also know that the total number of formulas is countably infinite, so there can not be an uncountable number of definable elements. And thus there must be an uncountable number of elements that are not definable by any formula. □

(b). Let f be an automorphism of $(\mathbb{R}, \cdot, +, <, 0, 1)$. Prove that the inverse image of any interval is itself an interval (i.e. the automorphism is continuous). Note that the f fixes all rational numbers (why?), and hence maps each interval with rational endpoints to itself. Finally use a proof by contradiction to show that f must map each real number to itself.

Proof. We know that any automorphism preserves order, and also that the inverse of an automorphism is another automorphism. Thus if $x \in (a, b)$, $f^{-1}(x) \in (f^{-1}(a), f^{-1}(b))$. Thus the inverse image of (a, b) is contained in $(f^{-1}(a), f^{-1}(b))$. Now we must check that the inverse image of (a, b) is all of $(f^{-1}(a), f^{-1}(b))$. Take some $y \in (f^{-1}(a), f^{-1}(b))$. Since f preserves order, $f(y)$ must be between $f(f^{-1}(a))$ and $f(f^{-1}(b))$, that is between a and b. Thus y is in the inverse image of the interval (a, b).

Now, for a contradiction, assume that for some real number r_1, $f(r_1) = r_2 \neq r_1$. Now take pick some interval, (q_1, q_2), with rational endpoints that includes r_2 but not r_1. Now $f^-1(r_2)$ must be contained in the inverse image of (q_1, q_2) which is just (q_1, q_2) by the preceding paragraph and the fact that f maps each rational number to itself. But this is a contradiction. □

Problem 5

Let A be an alphabet. Let A^* be the set of finite strings of symbols from A. Let W and W' be decidable subsets of A^*. Show that $W \cup W'$, $W \cap W'$, and $A^* \setminus W$ are all decidable.

Proof. Suppose P decides W, and suppose P' decides W'.

One can decide $W \cup W'$ as follows: Take input w and run P on w. If P says yes, then output "yes" and halt. If P says no then run P' on w and output "yes" and halt if P' says yes, and otherwise output "no" and halt.

One can decide $W \cap W'$ in a similar fashion. Take an input w, run both P and P' on w and output "yes" if both say yes otherwise output "no".

Finally one can decide $A^* \setminus W$ simply by running P and switching the output from yes to no and vice versa. □

Problem 6

Describe the decision procedure for the set of formulas of a language L, where L has a finite number of relation, function and constant symbols.

The point here is that the procedure like the definition will be recursive, and use the procedure P_t that decides whether an input is a term.

First the procedure is given an input, w, and it checks if w is of the form $w_1 = w_2$.

If so, it runs P_t to check whether w_1 and w_2 are terms. If so it says "yes" and halts. If not it says "no" and halts.

If w is not of the form $w_1 = w_2$, it checks whether w is of the form $R(w_1, \ldots, w_n)$ for some n-ary relation R, and if so it checks whether $w_1, \ldots, w_n$ are terms.

Next it checks whether w is of the form $w_1 \wedge w_2$. If so it runs itself of both w_1 and w_2 to check whether they are formulas. (Note that they are of shorter length, so this cannot cause an infinite loop.)

It does likewise for $\exists$, $\forall$, $\vee$, $\neg$, etc.