HOMEWORK 9: SOLUTIONS

Problem 1

(a). First we consider the model $(\mathbb{Z}, +)$. Is 1 a definable element in $(\mathbb{Z}, +)$? (Give a proof, of course.) List all the definable elements of $(\mathbb{Z}, +)$.

Proof. No, 1 is not a definable element. We have proven that definable elements are fixed by automorphism. It is relatively easy to check that multiplying by negative one is an automorphism of $(\mathbb{Z}, +)$ and clearly this does not fix 1. In fact, the only thing it fixes is 0, so that is the only element that could be definable.

Note that 0 is defined by the formula $\forall y(x + y = y)$. Thus we have shown that 0 is the only definable element. □

(b). Now consider the model $(\mathbb{Z}, +, <)$. Prove that each element of $(\mathbb{Z}, +, <)$ is definable. List all of the automorphisms of $(\mathbb{Z}, +, <)$

Proof. Now that we have $<$ as well, it is easy to see that 1 is definable. Intuitively, 1 is the least element greater than zero, and this can in fact be written in first order logic: $\neg\exists y(0 < y) \wedge (y < x)$. Note that I have used 0 in the formula. This is harmless as we have shown already that 0 is definable, and thus could be replace with the formula defining 0. □

Problem 2

Prove that there is a model $\mathfrak{Z}$ elementarily equivalent to $(\mathbb{Z}, +, <)$ where it is not the case that every element is definable.

Proof. The easiest way to do this problem is to use the Upward Lowenheim Skolem Theorem to construct a model elementarily equivalent to the integers but uncountable. Then one only has to observe that each definable element is defined by a different formula, and that there are only countably many formula.

There is a completely different proof, that I will also sketch. Let $\varphi_n(x)$ be the formula that defines n in $\mathbb{Z}$. But now consider some examples of things that $Th(\mathbb{Z})$ includes: $\forall x, y(\varphi_3(x) \wedge \varphi_5(y)) \to x < y$. Why is this? Well, just because $3 < 5$, and 3 and 5 are the only elements satisfying φ_3 and φ_5 respectively.

Now add a constant c to the language, and consider

$$\Phi := Th((\mathbb{Z}, +, <)) \cup \{\forall y(\varphi_1(y) \to y < c), \forall y(\varphi_2(y) \to y < c), \forall y(\varphi_3(y) \to y < c), \dots\}$$

In other words, c is greater than 1, 2, 3, etc.

The compactness theorem shows that Φ has a model, and this model has some element named by c. Dropping the constant c from the language, to return to $L = \{+, <\}$, we see that this model is elementarily equivalent to $(\mathbb{Z}, +, <)$. Now suppose that the element that was named by c (before c was dropped from the language) is named by some formula $\psi(x)$. Then for each n, $Th((\mathbb{Z}, +, <))$ implies the sentences

$$\forall x, y(\varphi_n(x) \wedge \psi(y) \to x < y)$$

and

$$\exists x \psi(x)$$

But this is impossible, since then these statements would also be true of $(\mathbb{Z}, +, <)$, and no such integer exists. $\square$

Problem 3

Let $W \subseteq A^*$. Show that the following are equivalent:

(1) W is enumerable.

(2) There is register machine program such that P on input w outputs the empty word and halts whenever $w \in W$, and P runs forever without outputing anything if w in not in W.

Proof. First (1) implies (2): Let P witness that W is enumerable. Now one can design a program that takes as input $w \in A^*$, than it begins running P, and stops only when P outputs w.

Now to check that (2) implies (1): Let P' witness (2). That is, $P' : w \mapsto \mathit{halt}$ iff $w \in W$. Now one can design a program that works at follows: as stage n it enumerates the first n elements of A^* and runs P' on each of them for n steps, outputting any w on which P' halts. Then it goes on to stage $n+1$. It is easy to see that such a program witnesses the enumerability of W.

$\square$

Problem 4

A set $W \subseteq A^*$ is called lexiographically enumerable iff there is a register machince program that outputs W is lexiographic order. Show that W is lexiographically enumerable iff W is decidable.

Proof. First decidable implies lexiographically enumerable: Suppose P decides W. Let P' send each element of A^* to P in lexiographic order, and output each element for which P says yes. Then P' lexiographically enumerates W.

Next lexiographically enumerable implies decidable: Case 1: W is infinite. Suppose that P' witnesses the lexiographic enumerability of W. Let P take input w. Next P begins running P'. If P' outputs w, P says yes. If P' outputs something that comes after w in lexiographic order, than P says no. Case 2: W is finite. Then W is decidable, since every finite set is decidable. $\square$

Problem 5

Let $A := \{a_0, a_1\}$ Carefully write out a register program that takes a string $w \in A^*$ and halts if the lenght of the string is divisible by three.

This is rather similar to an example from the book, and since no one had a lot of trouble with it, I'm not going to write a solution for it.

Problem 6

Suppose that $U \subseteq A^*$ is decidable, and suppose that $W \subseteq U$. Show that W is decidable if and only if both $U \setminus W$ and W are enumerable.

Proof. First note that it is clear that if U and W are decidable both $U \setminus W$ and W are enumerable. By problem 4, $U \setminus W$ is decidable, and decidable implies enumerable.

Now to show that if both $U \setminus W$ and W are enumerable, that W is decidable. Take an input w. First decide whether w is in U. If it isn't, output "no" and halt. If w is in U, then start running the enumeration procedure for both $U \setminus W$ and W. Since w is in U, one of these enumeration procedures will eventually list w. If it's the enumeration procedure for W, output"yes" and stop, otherwise output "no" and halt. □