SOLUTIONS: HOMEWORK 1

Section 1.1

Problem 2. Prove the second De Morgan Law: $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$

Proof. Say $x \in A\backslash(B \cap C)$ then x is in A and x is not in $B \cap C$. Hence x is in A but not in both B and C. Thus x is in either $A\backslash B$ or $A\backslash C$. We conclude $A\backslash(B \cap C) \subseteq (A\backslash B) \cup (A\backslash C)$.

On the other hand, say $x \in (A\backslash B) \cup (A\backslash C)$. Then x is in either $A\backslash B$ or $A\backslash C$. Thus x is either in A but not in B, or x is in A but not in C. So x is in A but can't be in both B and C, which means that $x \in A\backslash(B \cap C)$. We conclude $(A\backslash B) \cup (A\backslash C) \subseteq A\backslash(B \cap C$.

Therefore $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$.

□

Problem 8. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be given by $x \mapsto 1/x^2$. Then
(a) $f(E) = \{x \in \mathbb{R} : x \in [1/4, 1]\}$ for $E = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$.
and (b) $f^{-1}(G) = \{x \in \mathbb{R} : x \leq -1/2 \text{ or } x \geq 1/2\}$ for $G = \{x \in \mathbb{R} : 0 \leq x \leq 4\}$

Problem 21. Let $f : A \to B$ and $g : B \to C$ be functions and let H be a subset of C. Then we have $(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$.

Proof. First, we note that by definition $g^{-1}(H) = \{x \in B : g(x) \in H\}$. Thus $f^{-1}(g^{-1}(H)) = f^{-1}(\{x \in B : g(x) \in H\})$. Applying the definition again, we see that $f^{-1}(\{x \in B : g(x) \in H\} = \{y \in A : f(y) \in \{x \in B : g(x) \in H\}\}$. But considering what the last formulation of $f^{-1}(g^{-1}(H))$ is saying, we see that it is the set of y in A such that applying first f to y, and then g to $f(y)$ sends y into H. Thus we see that $f^{-1}(g^{-1}(H)) = \{y \in A : g(f(y)) \in H\}$, which in turn equals $\{y \in A : (g \circ f)(y) \in H\}$, which one notes is the definition of $(g \circ f)^{-1}(H)$. □

Section 1.2

Problem 2. $1^3 + 2^3 + 3^+ \ldots + n^3 = [n/2 \cdot (n + 1)]^2$ for all $n \in \mathbb{N}$

Proof. Let S be the set of $n \in \mathbb{N}$ such that the above statement is true. We confirm that 1 is in S by noting that $1^3 = [1/2 \cdot 2]^2$. Now assuming that $\{1, \ldots, k\} \subseteq S$, we see that $1^3 + \ldots + k^3 + (k+1)^3 = [k/2 \cdot (k+1)]^2 + (k+1)^3$. Beginning to simplify, we get that this equals $[1/4 \cdot (k^4 + 2k^3 + k^2)] + (k^3 + 3k^2 + 3k + 1)$, and then that it equals $1/4 \cdot (k^4 + 6k^3 + 13k^2 + 12k + 4)$. We can factor this as $1/4 \cdot (k^2 + 2k + 1)(k^2 + 4k + 4)$. But this last expression is clearly equal to $[1/2 \cdot (k + 1)(k + 2)]^2$. Thus $k + 1$ is in S, and we are done. □

Problem 7. $5^{2n} - 1$ is divisible by 8 for all n.

Proof. I had intended that this problem would be a proof by induction, but there is a much easier solution. Note $5^{2n} - 1 = 25^n - 1$. It is proven in the book that $a^n - b^n$ is divisible by $a - b$. Here we have $a = 25$ and $b = 1$, so $5^{2n} - 1$ is divisible by 24, and hence also by 8. □

Problem 14. $2^n < n!$ for all $n \in \mathbb{N}$ such that $n \geq 4$.

Proof. In this problem, we will use the second version of the Principle of Mathematical Induction. Let $n_0 = 4$. Clearly $2^4 = 16 < 24 = 4!$. Now pick an arbitrary $k \geq 4$. We want to show that if $2^k < k!$ then $2^{k+1} < (k+1)!$. Note that $2^k < k!$ implies that $2 \cdot 2^k < 2 \cdot k!$ □

Section 1.3

Problem 4. Exhibit a bijection between $\mathbb{N}$ and the odd integers greater than 13. Let $f(x) = 2x + 13$. It's pretty clear that this is a bijection, but to be complete, note that $g(y) = (y - 13)/2$ is an inverse function to f.

Problem 9. If S and T are denumerable, then so is $S \cup T$.

Proof. First note that one can obtain a bijection, $f_0 : \{x \in \mathbb{N} : x \text{ is even}\} \to S$, by composing a bijection between the even natural numbers and $\mathbb{N}$, with a bijection between $\mathbb{N}$ and S. In a similar fashion, one can obtain a bijection $f_1 : \{x \in \mathbb{N} : x \text{ is odd}\} \to T$.
Now define $f : \mathbb{N} \to (S \cup T)$ by $x \mapsto f_0(x)$ if x is even, and $x \mapsto f_1(x)$ if x is odd. Thus, f is a surjection of $\mathbb{N}$ onto S. □