SOLUTIONS: HOMEWORK 2

Section 1.3

Problem 10. The point of this problem was just to help you see the pattern that would lead to a solution to number 11, so I'm not going to spend any time justifying the solutions here. See the solution to number 11 for a proof that the answers are correct.

(a). 4

(b). 8

(c). 16

Problem 11. Show that if S_n has n-elements then $P(S_n)$ has 2^n.

Proof. Note that no matter which set of n elements that one chooses, the power set of that set will have the same size. Thus we may assume without loss of generality that $S_n := \{1, \dots, n\}$. The proof follows by induction:

Case $n = 1$: Then $P(S_n) = \{\{1\}, \varnothing\}$ which has size $2^1 = 2$.

Case $k + 1$: We assume that $|P(S_k)| = 2^k$ and consider $|P(S_{k+1})|$. Note that $P(S_{k+1})$ can be divided into two disjoint subsets consisting of (1) those subsets of S_{k+1} that contain the element $k + 1$ and (2) those that do not. Let's call the first set A and the second set B.

The collection of subsets of S_{k+1} that do not contain $k + 1$ is just $P(S_k)$, and by induction we know that this has size 2^k. Having determined the size of B, we would now like to determine the size of A.

Let $f : B \to A$ be given by $S \mapsto S \cup \{k + 1\}$. That is, f takes an element, S, of $P(S_k)$ (which is a subset of $\{1, \dots, k\}$) and adds the number $k + 1$ to the list of numbers that S contains. It is not hard to see that f is a bijection from B to A. Thus B also has 2^k elements.

Now we are done, for we have shown that $P(S_{k+1})$ has $2^k + 2^k = s \cdot 2^k = 2^{k+1}$ elements, as desired. □

Problem 12. The set of all finite subsets of $\mathbb{N}$ is countable.

Proof. First note that if $S \subset \mathbb{N}$ is finite then S has some largest element m. Thus, $S \subset \mathbb{N}_m := \{1, \dots, m\}$

Now we would like to show that $\mathbb{N}_m$ is countable. In fact, $\mathbb{N}_m$ is finite and has 2^m elements by the previous exercise.

Now we just apply Theorem 1.3.12 to $\bigcup_{m=1}^{\infty} \mathbb{N}_m$ to finish the proof. □

Section 2.1

Problem 2.

(i). $-(a+b) = (-a) + (-b)$

Proof. In class, I showed that $-a = -1 \cdot a$. Using this fact we see that $-(a+b) = -1 \cdot (a+b)$. Using the distributive law, we get $-1(a+b) = -1a + -1b$. Using the fact proved in class again, we get that $-1a + -1b = (-a) + (-b)$. □

(ii). $(-a)(-b) = ab$

Proof. I will use the above fact twice, and the associative and commutative properties of addition:

$$(-a)(-b) = (-1 \cdot a) \cdot (-1 \cdot b) = (-1 \cdot -1)ab = 1ab = ab$$

OK, looking at what I just wrote, I also used the fact that 1 is the multiplicative identity, and that $-1 \cdot -1 = 1$. The first is fine (it's property M3). The second needs to be justified.

First note that by the fact proven in class, $-1 \cdot -1 = -(-1)$. Next note that

$$1 = 1 + 0 = 1 + (-1 + -(-1)) = (1 + (-1)) + -(-1) = 0 + -(-1) = -(-1)$$

Thus $-1 \cdot -1 = 1$, which is all we needed to finish the proof.

□

(iii). $1/(-a) = -(1/a)$

Proof. Note that by Theorem 2.1.3 (a), all we need to show is that $-a \cdot -(1/a) = 1$. But,

$$-a \cdot -(1/a) = (-1 \cdot a) \cdot (-1 \cdot (1/a)) = (-1 \cdot -1) \cdot (a \cdot (1/a)) = 1 \cdot 1 = 1$$

□

(iv). $-(a/b) = (-a)/(b)$

Proof. By definition, $-(a/b)$ is the additive inverse of $(a \cdot (1/b))$, which we have proven is equal to $-1 \cdot (a \cdot (1/b))$. Thus it suffices to show that $(-a)/(b) = -1 \cdot (a \cdot (1/b))$. But this is clear, since

$$(-a)/b = (-a) \cdot (1/b) = -1 \cdot a \cdot (1/b)$$

□

Problem 4. Suppose $a \in \mathbb{R}$ and $a \cdot a = a$ then $a = 1$ or $a = 0$

Proof. Suppose that $a \neq 0$. Now one can apply Theorem 2.1.2 (b) with u and b equal to a, to show $a = 1$.

□

Problem 14. If $0 \leq a < b$, then $a^2 \leq ab < b^2$, but a^2 need not be less than ab.

Proof. First note that if $a = 0$, $a^2 = ab = 0$, so a^2 is not less than ab. But we do have $a^2 \leq ab < b^2$. Thus, we can now limit ourselves to the case $0 < a < b$, and this case is simply two application of Theorem 2.1.7 (c). Multiply both sides of $a < b$ by a to get $a^2 < ab$; multiply both sides by b to get $ab < b^2$. □

Problem 17. Take $a \in \mathbb{R}$ with $0 \leq a < \epsilon$ for all positive ϵ. Then $a = 0$

Proof. Although I didn't notice this when I assigned the problem, the proof of Theorem 2.1.9 works here as well. That is, assume not, and let ϵ equal $\frac{1}{2}a$. Then it is not the case that $a \leq \epsilon$. □