HOMEWORK 3

SECTION 2.1

Problem 20. If $0 < c < 1$, then $0 < c^2 < c$ and if $1 < c$ then $1 < c < c^2$.

Proof. In the first case, multiply both sides of $c < 1$ to get $c^2 < c$ as desired. In the second case, multiply both sides of $1 < c$ by c. (In both cases, one applies Theorem 2.1.7 c. ☐

Problem 21. There is no $n \in \mathbb{N}$ between 0 and 1.

Proof. Suppose not. Say $0 < n_0 < 1$. It is easy to see that for any natural number, m, other than 1, $m - 1$ is also a natural number. (A proof of this by induction is quite straightforward.) Thus, $n_0 - 1$ is a natural number, and $n_0 < 1$ implies $n_0 - 1 < 1 - 1 = 0$, which contradicts Theorem 2.1.8 c. ☐

Problem 26. Show that $a^{m+n} = a^m \cdot a^n$ and $(a^m)^n = a^{mn}$.

Proof. First we should be precise about what we mean by a^n. We define it inductively: $a^1 := a$ and $a^{k+1} := (a^k) \cdot a$.

Thus, by definition, $a^{m+1} = a^m \cdot a^1$. Now we assume that $a^{m+k} = a^m \cdot a^k$, and consider $a^{m+(k+1)}$. Clearly, we have that $a^{m+(k+1)} = a^{(m+k)+1}$ which by definition is $(a^{m+k}) \cdot a$. By our inductive hypothesis, this is $(a^m \cdot a^k) \cdot a = a^m \cdot (a^k)a$, which is $a^m \cdot a^{k+1}$, as desired.

Now we note that $(a^m)^1 = a^{m \cdot 1} = a^m$. Assume $(a^m)^k = (a^{mk})$. Now we have

$$(a^m)^{k+1} = (a^m)^k(a^m) = a^{mk} \cdot a^m = a^{mk+m} = a^{m(k+1)}$$

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SECTION 2.2

Problem 2. $|a + b| = |a| + |b|$ iff $ab \geq 0$.

Proof. Recall that $|c| = d$ iff $c^2 = d^2$. Using this fact we see that

$$|a + b| = |a| + |b| \text{ iff } a^2 + 2ab + b^2 = |a|^2 + 2|a||b| + |b|^2$$

Also note that $|a|^2 = a^2$, $|b|^2 = b^2$ and $|a||b| = |ab|$. Thus we can simplify $|a|^2 + 2|a||b| + |b|^2$ to $a^2 + 2|ab| + b^2$. Subtracting $a^2 + b^2$ from both sides, we see that

$$a^2 + 2ab + b^2 = |a|^2 + 2|a||b| + |b|^2 \text{ iff } 2ab = 2|ab|$$

But this happens iff $ab = |ab|$, which is just the same as saying $ab \geq 0$. ☐

Problem 5. $a < x < b$ and $a < y < b$ implies $|x - y| < b - a$

Proof. Without loss of generality, we may assume that $x > y$. Thus $|x - y| = x - y$, and we have to prove that $x - y < b - a$. Note that $x < b$ and $y > a$. Multiplying this last inequality by -1 gives -y<-a. So adding these two inequalities yields $x - y < b - a$, as desired.

I'm using the following fact: if $a < b$ and $c < d$, then $a + c < b + d$. I should probably prove this. We know that $b - a \in \mathbb{P}$ and so is $d - c$. Since $\mathbb{P}$ is closed under addition, we see that $b + d - (a + c)$ is in $\mathbb{P}$. But this just means that $a + c < b + d$. □

Problem 13. Determine and sketch . . .

Most of the difficulty that people had with this problem (those that had any difficulty at all) lay in getting the correct sketches. But adding graphs to the solutions is more trouble than it's worth, so I've just made the corrections on each individual homework.

Problem 14. Show that $V_\epsilon(a) \cap V_\delta(a)$ and $V_\epsilon(a) \cup V_\delta(a)$ are again neighborhoods of a.

Proof. Let's say that $\epsilon < \delta$. Then $V_\epsilon(a) \cap V_\delta(a)$ is the set of real numbers that are both within ϵ of a and within δ of a. But these are just the numbers that are within δ of a. That is $V_\epsilon(a) \cap V_\delta(a) = V_\delta(a)$.

Now consider $V_\epsilon(a) \cup V_\delta(a)$. This is the set of numbers that are either within ϵ of a or within δ. But clearly this just $V_\epsilon(a)$ □

SECTION 2.3

Problem 2. Let $S_2 := \{x \in \mathbb{R} | x > 0\}$. S_2 has a lower bound, and in fact 0 is the infimum of S_2. However, $\sup(S_2)$ does not exist.

Proof. First note that 0 is less than every positive number, and thus is a lower bound of S_2. Now assume that it is not the infimum of S_2. Then there a greater lower bound, say $l > 0$. But then $l/2$ is also in S_2 and is less than the supposed lower bound.

Now we have to show that there no upper bound. Again, this is an easy proof by contradiction: suppose $u > 0$ is an upper bound. But $2u > u$ and $2u$ is again in S_2. □

Problem 4. Let $S_4 := \{1 - \frac{-1^n}{n}\}$. Then $\sup S_4 = 2$ and $\inf S_4 = 1/2$.

Proof. Note that 2 and $1/2$ are both in S_4, and so the supremum has to be greater than or equal to 2 and the infimum has to be less than or equal to $1/2$. If the supremum of S_4 is greater than 2, then there must be some element of S_4 that is greater than 2. But if $1 + 1/n > 2$, then $1/n > 1$, and $n < 1$. But we have proven that all natural numbers are greater than or equal to 1.

Likewise if the infimum of S_4 is less than $1/2$ then there must be some element of S_4 that is less than $1/2$. First note that if $1 - \frac{-1^n}{n} < 1$ then n must be even. And if $1 - \frac{1}{n} < 1/2$, then $1/n > 1/2$ and $n < 2$. But n is also even, a contradiction. □