SOLUTIONS: HOMEWORK 4

Section 2.4

Problem 6. Let A and B be bounded nonempty subsets $\mathbb{R}$, and let $A + B := \{a+b|a \in A, b \in B\}$. Prove that $\sup(A+B)$ =$\sup(A)$+$\sup(B)$. And likewise with inf.

Proof. First note that for any $a \in A$ and $b \in B$, we have $a <\sup(A)$ and $b <\sup(B)$, and so $a+b <\sup(A)+\sup(B)$. Thus $\sup(A)+\sup(B)\}$ is an upper bound for $A+B$. It remains to show that it is a least upper bound.

Suppose there is a smaller upper bound, say u. Say $\sup(A) + \sup(B) - u$ is equal to ϵ. Then take $a \in A$ such that $\sup(A) - a < \epsilon/2$ and take $b \in B$ such that $\sup(B) - b < \epsilon/2$. Now we see that $a + b$ is closer than ϵ to $\sup(A) + \sup(B)$.

The proof for inf is similar. □

Problem 7. Let X be a nonempty set, and let f and g be defined on X and have bounded range in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x)|x \in X\} \leq \sup\{f(x)|x \in X\} + \sup\{g(x)|x \in X\}$$

and likewise with inf.

Proof. Apply the previous exercise to the image of f and the image of g. □

Problem 8. Let $X = Y := (0, 1)$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$. Define $f(x) := \sup\{h(x, y) : y \in Y\}$ and $g(y) := \inf\{h(x, y) : x \in X\}$. Then we have the following:

(a). $f(x) = 2x + 1$ and $textrminf\{f(x) : x \in X\} = 1$

Proof. Take any $x_0 \in (0, 1)$. First note that $2x_0 + 1$ is an upper bound to $\{2x_0 + y : y \in (0, 1)\}$ Now to prove that it is the least upper bound, we show that given any $V_\epsilon(2x_0 + 1)$ there is some element of $\{2x_0 + y : y \in (0, 1)\}$ contained in $V_\epsilon(2x_0 + 1)$. Let y be greater than $1-\epsilon$. Then $2x_0 + y$ is greater than $2x_0 + 1 - \epsilon$. Thus $2x_0 + y$ is an element of $V_\epsilon(2x_0 + 1)$. Since x_0 was any element in $(0, 1)$, we have $f(x) = 2x+1$ as desired.

Now to show that $\inf\{f(x) : x \in X\} = 1$. Note that $f(X)$ is the open interval (1,3), which has infimum 1, a fact which I think is sufficiently clear to be stated without proof. □

(b). $g(y) = y$ and $\sup\{g(y) : y \in Y\} = 1$

Proof. Fix y_0 in $(0, 1)$. Note that $2 \cdot 0 + y$ is a lower bound to $\{2x + y_0 : x \in X\}$. Reasoning as in part (a) we can show that it is, in fact, the infimum. It remains to show that $\sup\{g(y) : y \in Y\} = 1$. But since $g(Y) = (0, 1)$, this is clear. □

Problem 9. Let X, Y be as above. Let $h(x, y)$ be the function that assigns zero to (x, y) iff $x < y$ and otherwise equals one. Repeat (a) and (b) from above.

(a). Take $x_0 \in X$. Let $y_0 = x_0$. Then $h(x_0, y_0) = 1$, so $f(x_0) = 1$. Since x_0 was arbitrary, we see that $f(x) = 1$ for any $x \in X$and $\inf\{f(x) : x \in X\} = 1$.

(a). Take $y_0 \in Y$. Since $X = Y$ is an open set, one can take $x_0 \in X$ with $x_0 < y_0$. Then $h(x_0, y_0) = 0$. Thus $g(y)$ is always zero, and sup$g(Y) = 0$.

Problem 10. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f(x)$ and $g(y)$ be defined as above. Then $\sup(g(Y)) \leq \inf(f(X))$.

Proof. First we note that $\sup(g(Y))$ and $\inf(f(X))$ exist since h bounded implies f and g are bounded as well. To show that $\sup(g(Y)) \leq \inf(f(X))$ we have to show that for all y and for all x, $g(y) \leq f(x)$.

Assume for a contradiction that this does not occur, i.e. there is some x_0 and some y_0 such that $g(y_0) > f(x_0)$. Refering back to the definitions of $f(x)$ and $g(y)$ we see that this means that $\inf\{h(x, y_0) : x \in X\} > sup\{h(x_0, y) : y \in Y\}$. In particular it means that for any x_1, y_1 we have $h(x_1, y_0) > h(x_0, y_1)$. This has to hold true even when we pick our $x_1 = x_0$, and $y_1 = y_0$. But this is means that $h(x_0, y_0) > h(x_0, y_0)$, a contradiction. □

Section 2.5

Problem 4. In the proof of the second case of the Characterization Theorem show that x and y are in S.

Proof. We take z less than the sup of S, and claim that there are $x, y \in S$ such that $x < z < y$. The existence of y follows from the fact that z is less than the least upper bound of S, and hence not an upper bound on S. The existence of x follows from the fact that z is not a lower bound on S, as S is not bounded below. □

Problem 5. Suppose S is neither bounded above nor below, and has the property that whenever x, y are in S, then so is $[x, y]$. Then S is all of $\mathbb{R}$.

Proof. Take any z in $\mathbb{R}$. Then z is neither an upper or lower bound of S, and thus by definition there must be elements in S larger than z, and elements smaller than z. Let x and y be elements of S respectively smaller and larger than z. By hypothesis, $[x, y]$ in entirely contained in S, and thus so is z. But as z could have been any element of the real numbers, we see that $S = \mathbb{R}$. □

Problem 10. Let $\{I_n : n \in \mathbb{N}\}$ be a collection of nested intervals. Let us say that $I_n = [a_n, b_n]$, and let ξ, η be the supremum of the a_i's and the infimum of the b_i's respectively. In the book, it is shown that ξ is an element of $\bigcap_{n \in \mathbb{N}} I_n$. Now we will do the same for η: First note that η is greater than or equal to each a_i. Suppose not. Suppose $\eta < a_{i_0}$. Then there is b_k such that $\eta \leq b_k < a_{i_0}$. If $k \leq i_0$, then we have $b_k \geq b_{i_0} > a_{i_0}$, a contradiction. And if $k > i_0$, we have $b_k > a_k \geq a_{i_0}$, which again is a contradiction.

Now we have shown that given any interval $[a_i, b_i]$, both ξ and η are contained in that interval. But then any point that lies between ξ and η also lies in the interval $[a_i, b_i]$. Thus each interval, I_i, contains $[\xi, \eta]$, and thus $\bigcap_{n \in \mathbb{N}} I_n$ contains $[\xi, \eta]$.