

SOLUTIONS: HOMEWORK 5

SECTION 3.1

Problem 11. Let $x_n = (\frac{1}{n} - \frac{1}{n+1})$. Then $\lim(x_n) = 0$.

Proof. We have shown in class that if $y_n = \frac{1}{n}$ then $\lim(y_n) = 0$, and $z_n = \frac{1}{n+1}$ is the 1-tail of (y_n) , and thus also converges to zero. Thus $x_n = y_n - z_n$ also converges to zero. \square

Problem 13. Let $b \in (0, 1)$. Then if $x_n = nb^n$, then $\lim(x_n) = 0$.

Proof. Most of the class successfully followed the hint given in the problem. Since most of you understand this proof, I will give a different one here. Consider $y_n := \frac{x_{n+1}}{x_n} = \frac{(n+1)b^{n+1}}{nb^n} = \frac{n+1}{n} \cdot b$. By Theorem 3.2.11 (the Ratio Test), it will suffice to show that $\lim(y_n)$ exists and is less than 1. Since b is less than 1, it is enough to show that $\lim(\frac{n+1}{n}) = 1$. But $\frac{n+1}{n} = 1 + \frac{1}{n}$ and we know that $\lim(\frac{1}{n}) = 0$ so we are done. \square

SECTION 3.2

Problem 2. Let $x_n := (-1)^n$, and let $y_n := (-1)^{n+1}$. Then (x_n) and (y_n) diverge but $(x_n) + (y_n)$ and $(x_n)(y_n)$ both converge.

Proof. Both sequences alternate between one and negative one, and thus diverge. Note that $(x_n) + (y_n) = 0$ and $(x_n)(y_n) = (-1)$ so both the sum and the product converge. \square

Problem 7. If (b_n) is bounded and $\lim(a_n) = 0$ then $\lim(b_n a_n) = 0$.

Proof. Say (b_n) is bounded by $M \in \mathbb{R}$. Then $|b_n a_n| \leq M \cdot |a_n|$. Now apply Theorem 3.1.10. \square

SECTION 11.1

Problem 2. Show that (a, ∞) and (∞, b) are open, while $[a, \infty)$ and $(\infty, b]$ are closed.

Proof. Take x an element of (a, ∞) . Let $\epsilon := x - a$. Then $V_\epsilon(x) \subset (a, \infty)$, and thus it is open. One shows that (∞, b) is open in an analogous fashion.

Now note that the complement of $(\infty, b]$ is (b, ∞) , which we have just show is open. Thus, $(\infty, b]$ is closed. Likewise, one can show that $[a, \infty)$ is closed. \square

Problem 5. \mathbb{N} is a closed set.

Proof. To show that \mathbb{N} is closed, we will show that the complement of \mathbb{N} is open. Let $I_n = (n, n + 1)$ for $n \in \mathbb{N}$. Let $I_0 = (-\infty, 1)$. Note that the complement of the natural numbers in \mathbb{R} is the set $\bigcup_{n=0}^{\infty} I_n$. Since each I_n is an open interval, the union of the I_n is a union of open sets, which is itself open. \square

Problem 6. $A = \{1/n : n \in \mathbb{N}\}$ is not closed, but $A \cup \{0\}$ is closed.

Proof. Let B be the complement of A . Let $J_0 = (-\infty, 0]$. Let $J_n = (1/(n+1), 1/n)$. Let $J = \bigcup_{n=1}^{\infty} J_n$. Now we can write B as $(-\infty, 0] \cup J \cup (1, \infty)$.

To show that A is not closed, we can show that B is not open. Take any ϵ -neighborhood around 0. Then for n large enough, $1/n$ is in $V_\epsilon(0)$. But $1/n$ is A , not B , so every ϵ -neighborhood of zero contains points not in B . Thus B is not open.

To show that $A \cup \{0\}$ is closed, we can show that its complement is open. Since the complement of A is B , the complement of $A \cup \{0\}$ is $B \setminus \{0\}$. We can write $B \setminus \{0\}$ as $(-\infty, 0) \cup J \cup (1, \infty)$. Since this is a union of open intervals it is open. \square