SOLUTIONS: HOMEWORK 6

Section 3.3

Problem 7. Let $x_1 := 1$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Then (x_n) does not converge.

Proof. Suppose, for a contradiction, that (x_n) does converge to $a \in \mathbb{R}$. First we claim that a must be greater than zero. It is easy to see by induction that each term in the sequence is greater than or equal to 1, since x_1 is equal to 1, and x_{n+1} is greater than x_n.

The 1-tail of (x_n) also converges to a. But we can write each term of the 1-tail of (x_n) as a sum terms from the sequence (x_n) and the sequence $(1/x_n)$, since $x_{n+1} := x_n + 1/x_n$. Since a is not equal to zero, the sequence $(1/x_n)$ must converge to $1/a$. By our theorem about adding convergent sequences, we have that $\lim(x_{n+1})$ =$\lim(x_n)$+$\lim(1/x_n)$. Thus $a = a + 1/a$, implying that $0 = 1/a$, a contradiction.

□

Problem 8. Let $\{I_n : n \in \mathbb{N}\}$ be a set of nested, closed intervals. Then $\bigcap I_n$ is not empty.

Proof. Let $I_n := [a_n, b_n]$. Then (a_n) is an increasing sequence, and (b_n) is a decreasing sequence, with the property that $a_n \leq b_n$ for each n. We se that (a_n) is bounded below by a_1. Since $a_n \leq b_n \leq b_1$ for each n, we see that (a_n) is bounded about by b_1. Likewise we may show that (b_n) is a bounded sequence. Thus we may apply the Monotone Convergence Theorem to conclude that (a_n) converges to $a :=$sup$\{a_n : n \in \mathbb{N}\}$, and $\lim(b_n) = b :=$inf$\{b_n : n \in \mathbb{N}\}$.

Now we want to show that $a \leq b$. Suppose not, that is, suppose that $a > b$. Then a is not a lower bound for $\{b_n : n \in \mathbb{N}\}$. Thus, there is some b_k such that $b_k < a$. But note that b_k is an upper bound for $\{a_n : n \in \mathbb{N}\}$, since for any a_n, with $n > k$, $a_n \leq b_n \leq b_k$. But since a is the least upper bound, a cannot be greater than b_k, a contradiction.

□

Section 3.4

Problem 1. Let $x_{2n} := 0$ and let $x_{2n-1} = n$, so $X = (1, 0, 2, 0, 3, 0, \ldots)$. Then the entire sequence x_n is unbounded, but the subsequence (x_{2n}) is the constant sequence zero.

Problem 4.

(a). Let $x_n := (1 - (-1)^n + \frac{1}{n})$. Then (x_n) is divergent.

Proof. Note that it suffices to show that there are two subsequences of (x_n) that converge to different numbers. Thus we consider $y_n := x_{2n} = \frac{1}{2n}$ and $z_n := x_{2n-1} = 2 + \frac{1}{2n}$. We see that (y_n) converges to 0 and (z_n) converges to 2, and thus (x_n) cannot converge to anything. □

(a). Let $x_n :=$sin$n\pi/4$. Then (x_n) is divergent.

Proof. Again we define two subsequences of (x_n) such converging to different numbers. $y_n := x_{8n} = 0$ and $z_n := x_{8n+2} = 1$. Thus (x_n) diverges. □