SOLUTIONS: HOMEWORK 7

Section 3.5

Problem 6. Let p be a natural number. Let (x_n) be defined as follows. $x_1 := 1$ and $x_n + 1 := x_n + 1/n$. Then $\lim(|x_n + p - x_n|) = 0$ but (x_n) does not converge.

Proof. That the sequence does not converge is Example 3.5.6 (c). Now it only remains to not that $|x_{n+p} - x_n| = (\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+p}) < p\frac{1}{n}$, which converges to zero. □

Problem 8. A bounded, increasing sequence is Cauchy.

Proof. Suppose that (x_n) is increasing and bounded above by $M = sup\{x_n : n \in \mathbb{N}\}$. Suppose that (x_n) is not Cauchy. Then for some ϵ_0 for any $H \in \mathbb{N}$ one can find a pair m, n such that $|x_m - x_n| > \epsilon_0$. Since M is the supremum of $\{x_n : n \in \mathbb{N}\}$, we can find some x_k such that x_k is within ϵ_0 of M. Now take $m, n > k$ with $m > n$ such that $|x_m - x_n| > \epsilon_0$. But this means that $x_m > x_n + \epsilon_o > x_k + \epsilon_0 > M$, which is a contradiction. □

Problem 10. If $x_1 < x_2$ are arbitrary real numbers and $x_n := \frac{1}{2}(x_{n-2} - x_{n-1}$ then the sequence (x_n) converges to $x_1 + \frac{2}{3}a$

Proof. It might be useful to refer to Example 3.5.6 (a), which is very similar. This proof will simplify the argument given in the book by using Theorem 3.5.8. First we make the observation that each term in the sequence (after the first two) is the average of the previous two. This has the consequence that the distance from x_{n+2} to x_{n+1} is half the distance from x_{n+1} to x_n. Thus the series is contractive, and it converges.

Now that we know that (x_n) converges, we can pick a subsequence and find its limit. Let $a := |x_1 - x_2|$. It is easy to show by induction that $|x_{n+1} - x_n| = \frac{1}{2^{n-1}} \cdot a$. We write out the first few terms in the sequence, we'll notice a pattern, and then we'll use mathematical induction to confirm the pattern always holds. It will help you understand the following calculations if you draw x_1 and x_2 on a number line, and then draw a point half way between them, and another point halfway between x_2 and x_3, and so on.

$x_3 = x_1 + \frac{1}{2}a = x_2 - \frac{1}{2}a,$
$x_4 = x_3 + \frac{1}{4}a = x_2 - \frac{1}{4}a$
$x_5 = x_3 + \frac{1}{8}a = x_4 - \frac{1}{8}a$
$x_6 = x_5 + \frac{1}{16}a = x_4 - \frac{1}{16}a$
$x_7 = x_5 + \frac{1}{32}a = x_6 - \frac{1}{32}a$

The pattern which we notice, (again following Example 3.5.6 (a)), is in the odd terms. Note:

$x_3 = x_1 + \frac{1}{2}a$
$x_5 = x_3 + \frac{1}{8}a = x_1 + \frac{1}{2}a + \frac{1}{8}a$
$x_7 = x_5 + \frac{1}{32}a = x_1 + \frac{1}{2}a + \frac{1}{8}a + \frac{1}{32}a$

We guess that $x_{2n+1} = x_1 + \frac{1}{2}a + \frac{1}{8}a + \ldots + \frac{1}{2^{2n-1}}a$. We prove this by induction (by also noticing that $x_{2n+2} = x_{2n-1} + \frac{1}{2^{2n+2}}a$).

Now we recall (for instance, from Example 1.2.4 (f)) that for $r \neq 1$, $1 + r + r^2 + \ldots + r^n = \frac{1-r^{n+1}}{1-r}$. Applying this we see that

$$x_{2n-1} = x_1 + \frac{1}{2}a\left(1 + \frac{1}{4} + \ldots + \frac{1}{4^{n-1}}\right) = x_1 + \frac{1}{2}a\left(\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}\right) = x_1 + \frac{2}{3}a\left(1 - \frac{1}{4}^{n-1}\right)$$

and thus $\lim(x_{2n-1}) = x_1 + \frac{2}{3}a$.

□

SECTION 4.1

Problem 6. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$, and let $c \in I$. If there is a constant K such $|f(x) - L| \leq K|x-c|$ for $x \in I$ then $\lim_{x \to c} f(x) = L$

Proof. First note that c is a cluster point of I since $c \in I$ and I is an interval. Now take $\epsilon > 0$, let $\delta := \epsilon/K$. Then if $|x - c| < \delta = \epsilon/K$ then $|f(x) - L| < \epsilon$. □