EXAM 2

NAME:

TA:

Problem 1

(a). Let $g(x) = \ln(1 + e^{\cos x})$. Find $g'(x)$.

Solution. $g'(x) = 1/(1 + e^{\cos x})(e^{\cos x})(-\sin x)$

(b). Let $f(x) = a^x$. What is $f'(x)$? (Even if you happen to know the answer by heart, show the work of finding it.)

Solution. $\ln(f(x)) = \ln(a^x) = x \ln a$. Thus $D_x(\ln(f(x)) = D_x(x \ln a)$. So

$$\frac{1}{f(x)} f'(x) = \ln a$$

So $f'(x) = \ln a f(x)$ That is, $f'(x) = \ln a \cdot a^x$.

(c). Let $2y + e^x = e^{xy}$. Find $\frac{dy}{dx}$.

Solution. Using implicit differentiation, we see that $2\frac{dy}{dx} + e^x = e^{xy}(y + x\frac{dy}{dx})$. Thus

$$2\frac{dy}{dx} - e^{xy} x \frac{dy}{dx} = ye^{xy} - e^x$$

Thus $(2 - xe^{xy})\frac{dy}{dx} = ye^{xy} - e^x$ and

$$\frac{dy}{dx} = \frac{ye^{xy} - e^x}{(2 - xe^{xy})}.$$

Problem 2

(a). Let $f(x) = x^{1/2}$. Find a linear approximation to $f(x)$ near 25.

Solution. In the book's notation, $L(x) = f(a) + f'(a)(x - a)$. Here $f(a) = 25^{1/2}$ and $f'(a) = \frac{1}{2}25^{-1/2}$. So $L(x) = 5 + \frac{1}{10}(x - 25)$.

(b). Use your answer from part (a) to estimate the square root of 20.

Solution. To get estimate the square root of 20, we merely plug 20 into $L(x)$ above. We get

$$5 + \frac{1}{10}(20 - 25) = 4\frac{1}{2}$$

(c). Use Newton's Method on $y = x^2 - 20$ to estimate a square root of 20. Use 5 as your first approximation (x_0 in the book's terminology) and do Newton's method for two iterations (that is, find x_1 and x_2 in the book's terminology). Which approximation, x_2, or the answer from part (b) do you think is better?

Solution. $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - (x_n^2 - 20)/(2x_n)$. So

$$x_1 = 5 - 5/(2 \cdot 5) = 5 - \frac{1}{2} = 4\frac{1}{2} = \frac{9}{2}$$

And

$$x_2 = 4\frac{1}{2} - \frac{(\frac{9}{2})^2 - 20}{2 \cdot \frac{9}{2}} = 4\frac{1}{2} - \frac{\frac{81}{4} - \frac{80}{4}}{9} = 4\frac{18}{36} - \frac{1}{36} = 4\frac{17}{36}$$

Problem 3

Consider $f(x) = 3x^{5/3} - 5x$. Find (a) the critical points. Then, (b), find where the function has local minimums and local maximums, and find what the value of the function at these point or points are. Then, (c), find the intervals on which the function is increasing and decreasing, (d), find the inflections points, and (e) find where the function is concave up and concave down. Finally, (f), use this information to sketch a graph of the function. (Hint: remember that just as there is a positive and negative solution to $x^2 = c$, there is also a positive and negative solution to $x^{2/3} = c$.)

Solution. $f'(x) = 5x^{2/3} - 5$ and $f''(x) = \frac{10}{3x^{1/3}}$. So the critical points are 1 and -1, and the single point to check for inflection is 0, where f'' is not defined.

We see, plugging in $x = -8$ (getting $20 - 5 = 15$), $x = 0$ (getting -5) and $x = 8$ (getting $20 - 5 = 15$) that f is increasing to -1, decreasing between -1 and 1, and thereafter increasing. Thus, 1 is a local minimum and -1 is local maximum, by the first derivative test. The value of f at -1 is 2 and the value of f at 1 and -2.

Plugging in -1 (getting $-\frac{10}{3}$) and 1 (getting $\frac{10}{3}$) we see that f is concave down for negative x and concave up for positive x with an inflection point at 0.

Problem 4

We are making a cylindrical can without a top. We want the can to have a volume of 27π and we wish to have the surface area as small as possible. What should we chose as our radius and our height. If, in the course of doing this problem, you find a global minimum or maximum, be sure to explain how you know that it is a global minimum or maximum. (Recall that the volume of a cylinder of with a base of radius r and height h is $\pi r^2 h$, and that the surface of a cylinder without a top will be composed of a circle (the base) and a rectangle (the side).)

Solution. $V = 27\pi = \pi r^2 h$ so $h = 27/r^2$. The base has area πr^2 and the side has area equal to height times the circumference of the base $(2\pi r(27/r^2))$. So $A(r) = \pi r^2 + 2\pi r(27/r^2)$. Simplifying, we get that $A(r) = \pi r^2 + 2\pi \cdot 27r^{-1}$. Thus, $A'(r) = 2\pi r - 2 \cdot 27\pi r^{-2}$. Finding a common denominator, we see that $A'(r) = \frac{2\pi}{r^2}(r^3 - 27)$. Thus we obtain a critical point at $r = 3$. Now we may use the first or second derivative test to see that it is a global minimum.

For instance, to use the first derivative test, we note that the interval in question is $(0, \infty)$. The denominator of $A'(r)$ is always positive and the numerator is negative when $r < 3$ and positive when $r > 3$. Thus $A'(r) < 0$ for $r \in (0, 3)$ and $A'(r) > 0$ for $r \in (3, \infty)$. Thus 3 is a global minimum.

Now we plug 3 into $h = 27/r^2$ and get that h is also 3.

Problem 5

A 15 foot ladder is resting against a wall. The bottom is initially 12 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{3}$ ft/sec. How fast is the top of the ladder moving up the wall 9 seconds after we start pushing, and how high is it?

Solution. We have $x^2 + y^2 = 15^2$, where x is the distance to the wall and y is the height of the top of the ladder. Thus $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Since $\frac{dx}{dt}$ is $-1/3$, after 9 seconds, $x = 12 - \frac{1}{3} \cdot 9 = 12 - 3 = 9$ feet. (How did we know that $\frac{dx}{dt}$ is negative? Because x is getting smaller.)

Since $9^2 + y^2 = 15^2$, we get that $y^2 = 15^2 - 9^2 = 225 - 81 = 144 = 12^2$ Thus, the top of the ladder is 12 feet of the ground after 9 seconds, and (plugging into $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$) we see that $2 \cdot 9 \cdot \frac{-1}{3} + 2 \cdot 12 \cdot \frac{dy}{dt} = 0$. Thus the top of the ladder is moving upward at $\frac{1}{4}$ foot per second.