

**EXAM 3**

NAME:

TA and Section:

## PROBLEM 1

(a). Find the limit as  $x$  goes to 0 from the right of  $f(x) = x^{\sin x}$ .

First we note that this is in the form  $0^0$ , so we begin by taking the natural log of  $f(x)$  to convert it into a different form: We calculate the limit as  $x$  goes to zero from the right of

$$\ln(f(x)) = \sin x \ln x = \frac{\ln x}{\csc x}.$$

Now this is in the form  $\frac{\infty}{\infty}$  so we may apply L'Hopital's rule.

Differentiating numerator and denominator, we get

$$\frac{1/x}{-1(\sin x)^{-2} \cos x} = -\frac{\sin x \tan x}{x}.$$

Since

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

and  $\lim_{x \rightarrow 0^+} \tan x = 1$ , we see that  $\lim_{x \rightarrow 0^+} \ln(f(x)) = -1 \cdot 0 = 0$ . Thus

$$\lim_{x \rightarrow 0^+} f(x) = e^0 = 1.$$

(b). Evaluate the integral

$$\int_{-\sqrt{2}}^{\sqrt{2}} x \sin(x^2 + 2) dx$$

Let  $u = x^2 + 2$ . Then  $du = 2x dx$ . Now we can either figure out the new limits of integration now, or do the integral with unknown limits of integration, and change back from  $u$  to  $x^2 + 2$  at the end, and use our original limits of integration. First I will do the second method:

With the substitution our integral is

$$\frac{1}{2} \int_a^b \sin(u) du = \frac{1}{2} [-\cos(u)]_a^b = \frac{1}{2} [-\cos(x^2 + 2)]_{-\sqrt{2}}^{\sqrt{2}}$$

This turns out to be  $\frac{1}{2} [-\cos(4) - (-\cos(4))] = 0$ .

Now suppose that instead of resubstituting  $x^2 + 2$  for  $u$ , we change the limits of integration. We change  $\sqrt{2}$  to  $\sqrt{2}^2 + 2 = 4$ , and then we change  $-\sqrt{2}$  to  $(-\sqrt{2})^2 + 2 = 4$ . Thus we have

$$\frac{1}{2} \int_4^4 \sin(u) du,$$

but this is clearly 0 since the lower and upper limits of integration are the same.

## PROBLEM 2

Find the area of the region,  $R$ , bounded by the line  $y = x$  and the parabola  $y^2 = 8 + 2x$ . (Hint: before you do the problem you should sketch the line and the parabola, and decide whether it is better to integrate with respect to  $x$  or with respect to  $y$ .)

I will just describe the graph. The parabola  $y^2 = 8 + 2x$  can be rewritten as  $x = \frac{1}{2}y^2 - 4$ , so it is symmetric around the  $x$ -axis and has point most to the left at  $(-4, 0)$ . To find where the parabola intersects the line  $y = x$  we substitute  $y$  for  $x$  in  $y^2 = 8 + 2x$  getting  $y^2 = 8 + 2y$ . So  $y^2 - 2y - 8 = 0$  which we factor as  $(y - 4)(y + 2)$ . So the points of intersection are  $(4, 4)$  and  $(-2, -2)$ .

Once one sketches the graph, it should be clear that it is easier to integrate with respect to  $y$ . Thus we evaluate

$$\begin{aligned} \int_{-2}^4 [y - (\frac{1}{2}y^2 - 4)]dy &= [\frac{1}{2}y^2 - \frac{1}{6}y^3 + 4y]_{-2}^4 \\ &= \frac{1}{2}16 - \frac{1}{6}64 + 16 - (\frac{1}{2}4 + \frac{1}{6}8 - 8) = 8 - 10\frac{2}{3} + 16 - 2 - 1\frac{1}{3} + 8 \\ &= 32 - 14 = 18. \end{aligned}$$

## PROBLEM 3

Consider  $f(x) = xe^{-x}$ . Find where  $f(x)$  has local minimums and local maximums, and find what the value of the function at these points are. Find the intervals on which the function is increasing and decreasing. Find the inflections points, if there are any, and find where the function is concave up and concave down. Use L'Hopital's Rule to find the behavior as  $x$  goes to  $\infty$ . Finally, use this information to sketch a graph of the function.

First we will calculate  $f'(x)$  and  $f''(x)$ .

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

$$f''(x) = -e^{-x} - (D_x(xe^{-x})) = -e^{-x} - (1-x)e^{-x} = (-2+x)e^{-x}$$

Since  $e^{-x}$  is always greater than zero, we see that  $f'(x)$  is zero at  $x = 1$  and greater than zero for  $x < 1$  and less than zero for  $x > 1$ . Likewise,  $f''(x)$  is equal to zero at  $x = 2$  and greater than zero for  $x > 2$  and less than zero for  $x < 2$ .

Thus we see that that  $x$  is increasing to  $x = 1$  and decreasing thereafter. Thus there is a maximum at  $x = 1$ . This means that  $f$  is concave down at this point, and in fact the second derivative confirms that  $f$  is concave down until  $x = 2$ , where there is an inflection point, and concave up for  $x > 2$ .

It remains to find the behaviour of the function as  $x$  goes to infinity. As  $x$  goes to  $+\infty$  we see that the function is of the form  $\infty \cdot 0$ . Now we can either rewrite the function as

$$\frac{e^{-x}}{\frac{1}{x}} \text{ or as } \frac{x}{e^x}.$$

One may try the first way, but it will become quickly apparent that it will not lead to a solution. One the other hand, differentiating the numerator and denominator of  $\frac{x}{e^x}$  gives  $\frac{1}{e^x}$  which clearly goes to zero as  $x$  goes to  $+\infty$ .

As  $x$  goes to  $-\infty$  one see that  $f$  is of the form  $-\infty \cdot +\infty$ . Thus L'Hopital's rule is not needed.

Since some students took the instruction of the problem to mean that L'Hopital's rule should be used both for  $x$  going to  $+\infty$  and  $-\infty$ , it was decided to take no points off incorrect answers about the behaviour of  $f$  as  $x$  goes to  $-\infty$ .

## PROBLEM 4

Using Riemann sums, find the area under the graph of  $y = x + 1$  between 0 and 2. You may need to use one or more of the following:

$$\sum_{i=1}^n i = \frac{1}{2}n^2 + \frac{1}{2}n,$$

$$\sum_{i=1}^n i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,$$

$$\sum_{i=1}^n i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

We break up the interval from 0 to 2 into  $n$  pieces of equal length. (Equal length is not necessary, but makes the problem much easier). Thus the length of each one will be  $\frac{2}{n}$ . This is our  $\Delta x$ . The first piece will be  $[0, \frac{1 \cdot 2}{n}]$ , the second  $[\frac{1 \cdot 2}{n}, \frac{2 \cdot 2}{n}]$ , the third  $[\frac{2 \cdot 2}{n}, \frac{3 \cdot 2}{n}]$ , and so on. Thus  $x_i = \frac{i \cdot 2}{n}$ .

We wish to add up the area of the boxes with base  $[x_{i-1}, x_i]$  and height  $f(x_i)$ . (Here I am choosing to evaluate the function at the right endpoints. It is not necessary, but easier.) We write this sum as

$$\begin{aligned} \sum_{i=1}^n f(x_i) \Delta x_i &= \sum_{i=1}^n \left( \frac{i \cdot 2}{n} + 1 \right) \frac{2}{n} = \frac{2}{n} \sum_{i=1}^n \frac{i \cdot 2}{n} + \frac{2}{n} \sum_{i=1}^n 1 \\ &= \frac{4}{n^2} \sum_{i=1}^n i + \frac{2}{n} \sum_{i=1}^n 1 \end{aligned}$$

Using the first of the equations above, we see that the above equals

$$\frac{4}{n^2} \left( \frac{1}{2}n^2 + \frac{1}{2}n \right) + \frac{2}{n}n.$$

This simplifies to  $2 + \frac{2}{n} + 2$ . We are interested in seeing what happens as the boxes get thinner and thinner, that is, what happens as  $n$  goes to  $\infty$ . Clearly

$$\lim_{n \rightarrow \infty} \left( 2 + \frac{2}{n} + 2 \right) = 4.$$

## PROBLEM 5

A car begins at rest. It accelerates for 9 seconds at  $a(t) = \sqrt{t}$  meters/second<sup>2</sup>. How fast is it going after the 9 seconds of acceleration, and how far has it gone?

We know that velocity is an antiderivative of acceleration. Thus  $v(t) = \int \sqrt{t} dt = \frac{2}{3}t^{3/2} + C$ . Since the car starts at rest,  $v(0) = 0$  and thus  $C = 0$ . Likewise position is an antiderivative of velocity. So  $x(t) = \int \frac{2}{3}t^{3/2} = \frac{4}{15}t^{5/2} + C$ . Again, since we are only interested in how far the car travels in the first 9 seconds, we can call the place at which it starts  $x = 0$ . Thus  $x(0) = 0$  and again we see that  $C = 0$ . Now we plug in 9 to both equations, and get a velocity of  $\frac{2}{3}27 = 18m/s$  and a final position of  $\frac{4}{15}3^5 = \frac{4}{5}3^4 = \frac{4}{5}81 = 4 \cdot 16\frac{1}{5} = 64\frac{4}{5}m$ .