Exam 1 – Solutions

02/16/07

Problem 1

Find a general solution of $(2x^2+3)\frac{dy}{dx}+2xy=x$.

Solution:

First recognize that this is a linear differential equation and rewrite it as

$$\frac{dy}{dx}+\frac{2x}{2x^2+3}y=\frac{x}{2x^2+3},$$

observering that $P(x)=\frac{2x}{2x^2+3}$ and $Q(x)=\frac{x}{2x^2+3}$. Now we have that

$$\rho(x)=e^{\int\frac{2x}{2x^2+3}dx}=e^{\ln(2x^2+3)^{\frac{1}{2}}}=(2x^2+3)^{\frac{1}{2}},$$

from which we conclude that

$$(2x^2+3)^{\frac{1}{2}}y=\int\frac{x}{2x^2+3}(2x^2+3)^{\frac{1}{2}}dx=\int\frac{x}{(2x^2+3)^{\frac{1}{2}}}dx=\frac{1}{2}(2x^2+3)^{\frac{1}{2}}+C.$$

Thus, $y(x)=\frac{1}{2}+C(2x^2+3)^{-\frac{1}{2}}$

Problem 2

Find *all* solutions of $\frac{dy}{dx} = (\cos x)(2y-1)^2$

Solution:

First recognize that this is a separable equation, and remember to find both the general solution and check for singular solutions.

The general solutions is obtained by writing $\int (2y-1)^{-2} dy = \int \cos x dx$. Thus,

$$\frac{1}{2(2y-1)^{-1}} = \sin x + C$$

$$\text{and } 4y - 2 = \frac{-1}{\sin x + C}$$

$$\text{and } y = \frac{1}{4}\left(\frac{-1}{\sin x + C} + 2\right).$$

Now we have to find the singular solutions. Suppose r is a root of $(2y-1)^2$, that is, let $r = 1/2$. Then $y = 1/2$ can be seen to be a solution to the differential equation, since $d(1/2)/dx = 0 = (\cos x)(2\frac{1}{2} - 1)^2$. Moreover, it is easy to see that $y = \frac{1}{2}$ can not be obtained from the general solution by choosing some value of C..

Problem 3

For both of the following pairs of functions, either show that they are linearly independent or find $a, b \in \mathbb{R}$ not both zero[1] such that $af_1 + bf_2 = 0$.

(a): $f_1(x) = 0, f_2(x) = e^x$

Solution:

The best way to do these problems is to compute the Wronskian, and then if one finds linear dependence, figure out the a, b.

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} 0 & e^x \\ 0 & e^x \end{vmatrix} = 0$$

So we see that they are linearly dependent, and it is easy to see that we may let a be any nonzero real number, and let $b = 0$.

Of course, in this situation we could skip computing the Wronskian, and see immediately that by choosing and a and b as above that we would demonstrate linear dependence.

(b): $f_1(x) = xe^x, f_2(x) = x^2e^x$

Solution:

Again, we compute the Wronskian:

$$\begin{vmatrix} xe^x & x^2e^x \\ (1+x)e^x & (2x+x^2)e^x \end{vmatrix} = (2x^2 + x^3)e^{2x} - (x^2 + x^3)e^{2x} = x^2e^{2x}.$$

This is not equal to zero, so f_1 and f_2 are linearly independent.

[1] the "not both zero" was omitted from the exam in error

Problem 4

For both of the following differential equations, there is a substitution that make the equation either separable or linear. If it is the former, perform the substitution and rewrite the differential equation in the form $f(v)dv = g(x)dx$. If it is the latter, perform the substitution and rewrite the equation in the form $\frac{dv}{dx} + P(x)v = Q(x)$. (You do not have to solve the differential equation.)

(a): $(x^2+1)\frac{dy}{dx} + 2xy = (x^2+x)y^{3/5}$

Solution:

First we note (tipped off perhaps by the $y^{3/5}$) that this is a Bernouli equation. Thus we write it as

$$\frac{dy}{dx} + \frac{2x}{(x^2+1)}y = \frac{(x^2+x)}{(x^2+1)}y^{3/5},$$

and we use the substitution $v = y^{1-3/5} = y^{2/5}$. Thus $y = v^{5/2}$ and $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{5}{2}y^{3/2}\frac{dv}{dx}$. Thus,

$$\frac{5}{2}y^{3/2}\frac{dv}{dx} + \frac{2x}{(x^2+1)}v^{5/2} = \frac{(x^2+x)}{(x^2+1)}y^{3/2}.$$

Dividing each side by $\frac{5}{2}y^{3/2}$ yields

$$\frac{dv}{dx} + \frac{4x}{5(x^2+1)}v = \frac{2(x^2+x)}{5(x^2+1)}.$$

This is the linear equation we seek.

(b): $y^3\frac{dy}{dx} = x^3 + 2xy^2$

Solution:

It is not hard to see that this equation is homogeneous. Thus we divide each side by y^3 to get

$$\frac{dy}{dx} = \frac{x^3+2xy^2}{y^3} = \left(\frac{y}{x}\right)^{-3} + 2\left(\frac{y}{x}\right)^{-1}.$$

Now we make the subsitution $v = \frac{y}{x}$, $y = vx$, and therefore $\frac{dy}{dx} = v + x\frac{dv}{dx}$ to get

$$v + x\frac{dv}{dx} = (v)^{-3} + 2(v)^{-1}.$$

$$\text{Thus, } x\frac{dv}{dx} = \frac{1+2v^2-v^4}{v^3}$$

$$\text{and } \frac{v^3}{1+2v^2-v^4}dv = \frac{1}{x}dx$$

Problem 5

A rumour spreads through a city of $M = 100,000$ people. Let P be the total number of people in the city who have heard the rumor. Assume that the percentage of the population that has heard the rumour grows at a rate proportional to $(M - P)$.

(a): Write an differential equation describing $\frac{dP}{dt}$.

Solution:

The correct solution is very straightforward. The percentage of the population who have heard the rumor grows proportional to $(M - P)$. So $\frac{d(percentage)}{dt} = k(M - P)$. Thus the number of people who have heard the rumor is equal to the percentage of people times M, the total number of people. Thus $\frac{dP}{dt} = Mk(M - P)$. Or if you prefer to absorb M into the constant k, $\frac{dP}{dt} = k(M - P)$.

However, I had hinted that you should look at how the book describes the spread of rumors, and, indeed, I meant to set of the problem so that the correct solution would agree with the book's example about rumors. So only 1 point was taken off for those who said $\frac{dP}{dt} = Pk(M - P)$.

(b): Suppose that at $t = 0$, there are $1,000$ people who have heard the rumour, and that this number is growing at a rate of $1,000$ people per day. Write an equation for $P(t)$.

Solution:

Again, the correct solution is very straightforward: It is a separable equation, so one has

$$\int \frac{dP}{M - P} = \int k dt.$$

So $-\ln(M - P) = kt + C$, and we see $1/(M - P) = Ce^{kt}$. Thus

$$P = M - \frac{1}{C} e^{-kt}.$$

From the initial conditions and $\frac{dP}{dt} = Mk(M - P)$, we obtain that $1,000 = 100,000k(100,000 - 1,000)$ and thus $k = 1,000/(100,000 \cdot 99,000) = 1/9,900,000$ (Obviously, if you were using $\frac{dP}{dt} = k(M - P)$ you will get a different k.) Again from the initial conditions, you get $1,000 = 100,000 - \frac{1}{C}$, and thus $C = \frac{1}{99,000}$.

On the other hand, if you used the logistic equation, $\frac{dP}{dt} = Pk(M - P)$, in part (a) the answer is also quite straightforward as long as you remember the solution of the logistic equation, namely

$$P(t) = \frac{MP_0}{P_0 + (M - P)e^{-kMt}}$$

Then we have

$$P(t) = \frac{100,000 \cdot 1,000}{1,000 + (99,000)e^{-k100,000t}}$$

So it remains only to find k. But our intial conditions also tell us that

$$1,000 = k1,000(100,000 - 1,000).$$

So $k = \frac{1}{99,000}$.