Exam 2

02/16/07

Problem 1

One solution of the differential equation $y^{(3)} - 7y'' + 15y' - 9y = 0$ is e^t. Find the general solution.

Solution:

First we note that the characteristic equation associated to the differential equation is $r^3 - 7r^2 + 15r - 9 = 0$. We want to find the roots $r_1, \ldots, r_3$ of this equation. Since we know that e^t is a solution of the characteristic equation, we see that $r = 1$ is a root of the equation and thus $r - 1$ divides it.

So we write the $r^3 - 7r^2 + 15r - 9 = (r-1)(r^2 - 6r + 9)$. Now it is easy to see the characteristic equation is equal to $(r-1)(r-3)(r-3)$. Thus the general solution is $c_1 e^t + (c_2 + c_3 x)e^{3t}$.

Problem 2

(a)

What should you try for a particular solution for the equation $y'' + 2y' + 2 = t^2e^{4t} + \cos t$ when using the method of undetermined coefficients? You do NOT actually have to find the particular solution.

Solution:

First one has to find a the collection of derivatives of terms of $t^2e^{4t} + \cos t$: $\{t^2e^{4t}, te^{4t}, e^{4t}, \sin t, \cos t\}$.

Next one has to find the solution of $y'' + 2y' + 2 = 0$: in this case, $c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1, r_2 = \frac{-2\pm\sqrt{4-8}}{2}$. This is only to check that there is no linear combination $t^2e^{4t}, te^{4t}, e^{4t}, \sin t, \cos t$ that is a solution of $y'' + 2y' + 2 = 0$. There isn't, so the particular solution that one should try is $At^2e^{4t} + Bte^{4t} + Ce^{4t} + D\sin t + E\cos t$.

(b)

What should you try for a particular solution for the equation $y'' - 8y' + 16 = t^2e^{4t} + \cos t$ when using the method of undetermined coefficients? You do NOT actually have to find the particular solution.

Solution:

Again, one needs to find the solutions to the associated homogeneous equation. This time the general solution is $(c_1 + c_2x)e^{4t}$, and we see that whenever $A = D = E = 0$, the particular solution found in part (a) is also a solution to $y'' + 2y' + 2 = 0$. Thus we try $t^2(At^2e^{4t} + Bte^{4t} + Ce^{4t}) + D\sin t + E\cos t$ instead.

Problem 3

Two solutions of

$$y'' + \frac{1}{t}y' + \frac{1}{t^2}y = 0$$

are $y_1 = t$ and $y_2 = t \ln t$. Use variation of parameters to find a particular solution to

$$y'' + \frac{1}{t}y' + \frac{1}{t^2}y = \frac{1}{t}.$$

Solution:

If $y_1 = t$ and $y_2 = t \ln t$ are solutions to $y'' + \frac{1}{t}y' + \frac{1}{t^2}y = 0$ then

$$y_p(t) = y_2(t) \int \frac{y_1(t)f(t)}{W(t)} - y_1(t) \int \frac{y_2(t)f(t)}{W(t)}$$

where $W(t)$ is the Wronskian, and f(t) in this case is $1/t$.

Note that $y_1'(t) = 1$ and $y_2'(t) = \ln t + 1$. Thus $W(t) = t \ln t + t - t \ln t = t$

So we have

$$y_p(t) = t \ln t \int \frac{1}{t} - t \int \frac{\ln t}{t} = t(\ln t)^2/2$$

Problem 4

A block of 8 lbs is attached to the end of a spring, causing it to stretch $\frac{1}{2}$ foot. The block is then pulled down $\frac{1}{4}$ foot and released. Assume that 8 lbs of weight is 1 slug of mass. Assume that there is damping of 1 lbs-sec/ft. Determine the motion of the block. Is the motion underdamped, critically damped, or over damped?

Solution:

We have $F = kx$, so $8 = k\frac{1}{2}$, and $k = 16$. We also have $m = 1$, and $c = 1$. Thus we have the equation $x'' + x' + 16x$, with intial conditions $x(0) = -\frac{1}{4}$ and $x'(0) = 0$.

Clearly, the motion is underdamped, as $c^2 = 1 < 4km = 64$. Thus we expect to find that the two roots of the characteristic equation are complex numbers of the form $a+bi$, $a-bi$. One uses the quadratic formula to find that this is indeed the case and $a = -\frac{1}{2}$ and $b = \frac{\sqrt{63}}{2}$. So $x(t) = e^{-\frac{1}{2}t}(A\cos\frac{\sqrt{63}}{2}t + B\sin\frac{\sqrt{63}}{2}t)$. Plugging in 0 one gets $A = -\frac{1}{4}$.

Finally, we calculate $x'(t)$:

$$-\frac{1}{2}e^{-\frac{1}{2}t}(-\frac{1}{4}\cos\frac{\sqrt{63}}{2}t + B\sin\frac{\sqrt{63}}{2}t) + e^{-\frac{1}{2}t}(\frac{1}{4}\frac{\sqrt{63}}{2}\sin\frac{\sqrt{63}}{2}t + B\frac{\sqrt{63}}{2}\cos\frac{\sqrt{63}}{2}t)$$

Plugging in 0, we get

$$-\frac{1}{2}1(-\frac{1}{4}1 + B0) + 1(\frac{1}{4}\frac{\sqrt{63}}{2}0 + B\frac{\sqrt{63}}{2}1) = \frac{1}{8} + \frac{\sqrt{63}}{2}B = 0.$$

Thus $B = \frac{1}{4\sqrt{63}}$.