Exam 3

04/20/07

Problem 1

Consider $y'' + \lambda y = 0$ where furthermore $y(0) = y'(\pi/2) = 0$. Find the eigenvalues and eigenfunctions. (You may assume that there are no negative eigenvalues.)

Solution:

First let $\lambda = 0$. Thus the equation is $y'' = 0$ with general solution $Ax + B$. $y(0) = 0$ implies that $0 + B = B = 0$. Also $0 = y'(\pi/2) = A$. Thus $\lambda = 0$ is not an eigenvalue.

Now let $\lambda = \alpha^2$. The general solution is $A \cos \alpha x + B \sin \alpha x$. Plugging in 0 we see that $A = 0$, and taking the derivative gives us that $y'(x) = B\alpha \cos \alpha x$. Plugging in $\pi/2$ shows us that $\alpha \in \{1, 3, 5, 7, \dots\}$.

So the eigenvalues are n^2 for n odd, and the eigenfunctions are $\sin nx$ for n odd.

Problem 2

Part (a)

Let $f(t) = \sin t$ for t between 0 and π, and $f(t) = 0$ for t between π and 2π. The cosine series for $f(t)$ begins

$$\frac{1}{\pi} - \frac{4}{\pi}(-\frac{1}{3}\cos\frac{t}{2} + \frac{1}{5}\cos\frac{3t}{2} + \frac{2}{12}\cos\frac{4t}{2} + \frac{1}{21}\cos\frac{5t}{2} + \dots)$$

and the sine series for $f(t)$ begins

$$\frac{1}{2}\sin t - \frac{4}{\pi}(\frac{1}{3}\sin\frac{t}{2} + \frac{1}{5}\sin\frac{3t}{2} - \frac{1}{21}\sin\frac{5t}{2} + \dots).$$

Let $g(t) = \cos t$ for t between 0 and π, and $g(t) = 0$ for t between π and 2π. Find the first 4 terms of the sine and cosine series for $g(t)$. Hint: no integration is needed.

Solution:

Here all one needs to remember is that given the Fourier series for $f(t)$ one can obtain the Fourier series for $f'(t)$ by differentiating the Fourier series for $f(t)$ termwise. Note that if one extends $g(t)$ to an odd function, then it is the derivative of $f(t)$ extended to an even function. Thus one obtains the sine series of for $g(t)$ by differentiating termwise the cosine series of $f(t)$.

Likewise, one obtains the cosine series of for $g(t)$ by differentiating termwise the sine series of $f(t)$.

Part(b)

Let $h(t)$ be the function that is 0 between 0 and 2 and 20 between 2 and 4. Extend this to be an odd function of period 8, and find its Fourier series.

Solution:

Since we extend $h(t)$ to an odd function, we are calculating the sine series for $h(t)$.

$$b_n = \frac{2}{4}\int_0^4 h(t)\sin\frac{n\pi}{4}t\,dt = \frac{1}{2}\int_2^4 20\sin\frac{n\pi}{4}t\,dt =$$

$$-10\frac{4}{n\pi}[\cos\frac{n\pi}{4}t]_2^4 = -10\frac{4}{n\pi}[\cos n\pi - \cos(n\pi/2)].$$

Note that for n odd, $\cos n\pi$ is -1 and $\cos(n\pi/2)$ is 0. For n equal 2 mod 4, one gets 1 and -1 for $\cos n\pi$ and $\cos(n\pi/2)$ respectively, and n a multiple of four, one gets 1 and 1. So $b_n = \frac{40}{n\pi}$ for n odd, $b_n = \frac{-80}{n\pi}$ for $n \in \{2, 6, 10, 14, \dots\}$, and $b_n = 0$ for n a multiple of 4.

Problem 3

Find a steady periodic solution to $x'' + 2x = h(t)$ where $h(t)$ is the function from Problem 2, Part (b).

Solution:

This should have read, "where h(t) is the function from Problem 2, Part (b) after it has been extended to an odd function".

First one writes $x'' + 2x = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{4}t)$ where the b_n's are those calculated in the previous problem. Now one looks for a solution of the form $x_{sp}(t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{4}t)$.

Thus one has

$$-\sum_{n=1}^{\infty} c_n \frac{n^2\pi^2}{4^2} \sin(\frac{n\pi}{4}t) + 2\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{4}t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{4}t)$$

Thus $c_n = b_n/(2 - \frac{n^2\pi^2}{4^2})$.

Problem 4

Suppose that two metal rods (with thermal diffusivity k=1 m^2/s) of length 2 meter are joined end to end. One begins at 20 degrees and the other at 0. The rods are insulated laterally, and their ends are embedded in ice. Find $u(x,t)$. Hint, if the cold rod is on the left, you can use Problem 2, Part b.

Solution:

One may simply use Theorem 1 from 9.5 to say

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2}{4^2}t} \sin(\frac{n\pi}{4}t)$$

where the b_n are again as in Problem 2 (b).