EXAM 1: TAKEHOME

Problem 1

Define a number, x, to be *algebraic* iff there is some polynomial $x^n + q_{n-1}x^{n-1} + q_{n-2}x^{n-2} + \ldots + q_0 = 0$ with each $q_i \in \mathbb{Q}$. For example, $\sqrt{2}$ is a algebraic number, as is any rational number, but π is not. Show that there are countably many algebraic numbers. (Feel free to use without proof the fact that any polynomial in one variable has a finite number of solutions.)

Problem 2

(a). Let $p\mathfrak{n}q$ stand for "neither p nor q". That is, $p\mathfrak{n}q$ is true when, and only when, both p and q are false. Show that every formula in propositional logic is equivalent to one only using $\mathfrak{n}$.

(b). Suppose that p stands for "There are tickets." and q stands for "I will go to the game." Use your newfound skill with $\mathfrak{n}$ to say (i) $\neg q$, and (ii) $p \to q$ in English using only "neither ... nor" as a logical connective.

Problem 3

Consider the following structure: $(\mathbb{R}, +, \cdot, <, f)$ where $+, \cdot, <$ are as normal, and f is a unary function. Express the following in first order logic (in the language of $+, \cdot, <, f$):

(a). Every positive number has a square root.

(b). If f is strictly monotone then f is injective.

(c). f is continuous.

Problem 4

(a). Find a finite collection of sentences in some language such that there is a finite model of size 1 through size n but no larger models.

(b). Find a finite collection of sentences in some language such that there is a finite model of any even size, but no finite model of any odd size.

(c). Find a finite collection of sentences in some language such that there is a finite model of size n^2 for any n but no finite models of size m for m not a square.

(d). Find a finite collection of sentences in some language such that there is a finite model of size p for each prime p, but no other finite models..

Problem 5

Fix a language L. Fix a model $\mathfrak{M}$ in the language L. Suppose that a relation R is definable in $\mathfrak{M}$, say by $\varphi(x_1, \ldots, x_n)$. Now suppose that one expands the language to $L' := L \cup R$. Let $\mathfrak{M}'$ be identical to $\mathfrak{M}$ except that it includes the new relation R. Show that for every formula σ in L' there is a formula θ using only the symbols from L such that $\mathfrak{M}' \models \sigma \leftrightarrow \theta$.

Problem 6

(a). Consider $\mathfrak{M} := ([0,1], <)$. Show that 0 and 1 are definable in $\mathfrak{M}$, that is, show that the set containing only 0 is definable, and that the set containing only 1 is definable.

(b). Show that any automorphism of a model maps any definable set to itself.

(c). Show that 0 and 1 are the only definable elements in $\mathfrak{M}$