EXAM 2: TAKEHOME

Problem 1

We will define a collection, $\mathfrak{K}$, of models in a given language to be a Δ-elementary class iff there is a collection of sentences Φ such that the models contained in $\mathfrak{K}$ are precisely the models of Φ. For example, the collection of models $\mathfrak{M}$ in the language $L = \{e, \circ\}$ that form are groups is a Δ-elementary class: the group axioms are expressible in first order logic and these axioms form the set of sentences Φ.

Hopefully (a) and (b) will seem easy once you internalize the definition. For (a), (b), and (c), let $L := \{<\}$:

(a). Let $\mathfrak{K}_1$ be the collection of linear orders. Is $\mathfrak{K}_1$ an Δ-elementary class?

(b). Let $\mathfrak{K}_2$ be the collection of finite linear orders. Is $\mathfrak{K}_2$ a Δ-elementary class?

(c). Let $\mathfrak{K}_3$ be the collection of well-orders. (A well-order is a linear order with no infinite descending chain of elements. For example $(\mathbb{N}, <)$ but not $(\mathbb{Q}, <)$ or $(\mathbb{Z}, <)$.) Is $\mathfrak{K}_3$ a Δ-elementary class?

(d). Let L be any language and let $\mathfrak{M}$ be any infinite model in that language. Is the set of models isomorphic to $\mathfrak{M}$ a Δ-elementary class?

Problem 2

Let $\mathfrak{K}$ be a Δ-elementary class. Let $\mathfrak{K}' \subseteq \mathfrak{K}$ be the class of models in $\mathfrak{K}$ that are infinite. Show that $\mathfrak{K}'$ is also Δ-elementary.

Problem 3

Fix a language, L, and a complete theory T in L. Let S be a collection of L formulas, all with free variable x. Say that S is consistent iff for all finite collection $\varphi_1(x), \ldots, \varphi_n(x)$ taken from S one has that T proves $\exists x(\varphi_1(x) \wedge \ldots \wedge \varphi_n(x))$. We define a *type* to be a consistent set of formulas with the same free variable.

(a). Let our language, L, be $\{+, \cdot, 0, 1\}$. Recall that if r is an element of $\mathfrak{R}$, a model elementarily equivalent to the real numbers, we have defined what it means for r to be infinitessimal. Show that "infinitessimally close to zero and positive" can be expressed as a type. That is, show that there is a consistent set of formulas $p(x)$, each with free variable x, such that each formula in $p(x)$ is true of r whenever r is infinitessimal greater than zero. Moreover, show that if r is an element of which every formula in the type is true then r is infinitessimal and greater than zero.

(b). Let $\mathfrak{M}$ be any model. Let $p(x)$ be a type. Show that the set of elements m in $\mathfrak{M}$ of which $p(x)$ is true is fixed by automorphism.

PROBLEM 4

(a). Fix a register machine P. Consider the set of pairs (x, y) where P takes input x and outputs y. Is this set of pairs enumerable? Is it always decidable, never decidable, or does its decidability depend on P?

(b). Fix a pair (x, y). Let S be the set of programs P that output y when given input x. Is this set enumerable? Is it always decidable, never decidable, or does it's decidability depend on the pair x, y? (To be precise, I should ask this question not of S, but of $S' = \{w_P : P \in S\}$ where w_P, as always, stands for the source code of the register program P.)

PROBLEM 5

Let W be the set of w_P such that P gives the same output whenever it halts (no matter what the input was). Is W decidable? Is it enumerable?

PROBLEM 6

Say a register machine P' *extends* P iff on any input on which P halts, so does P' and, moreover, they both give the same output. (In other words, if you think of programs as partial functions from the set of inputs to the set of outputs, P' extends P in the sense of functions.)

Let P_0 be the program that given w_P, runs P until it halts, and then outputs the number of steps that it took P to halt (and P_0 never halts if P does not halt). Show that there is no extension P' of P_0 that always halts.