FINAL: TAKEHOME

DUE MAY 12TH, ROOM 250 ALTGELD, IN MY MAILBOX BY 4:00 PM

PROBLEM 1

Prove that it is not the case that every sentence in propositional logic is equivalent to one using only $\wedge$ and $\rightarrow$.

PROBLEM 2

Let A and B be disjoint infinite sets. Define $|A| + |B|$ as $|A \cup B|$. Suppose that $|A| \geq |B|$. Then it is a fact that $|A| + |B| = |A|$. Prove this fact using the simplifying assumption that each of A and B are countable or equal in size to the reals.

PROBLEM 3

Let A and B be infinite sets. Let $A_b := \{(a, b) | a \in A\}$. So for each $b \in B$, there is a bijection between A_b and A, and for each b_1, b_2 distinct elements of B, $A_{b_1} \cap A_{b_2}$ is empty.

Define $|A| \cdot |B|$ as $|\bigcup_{b \in B} A_b|$. Again, it is a fact that $|A| \cdot |B| = max\{|A|, |B|\}$.

(a). Prove this fact using the simplifying assumption that each of A and B are countable or equal in size to the reals.

(b). Using the same simplifying assumption, show that $|A| \cdot |B| = |B| \cdot |A|$. (This should will be easy after part (a)).

PROBLEM 4

Fix a language L, a model $\mathfrak{M}$ with universe M, and a formulas $\varphi_1(x_1, \ldots x_n)$ and $\varphi_2(x_1, \ldots x_n)$. Prove the following facts, in detail, and from the definitions.

(a). The subset of M^n defined by $\varphi_1(\vec{x}) \wedge \varphi_2(\vec{x})$ is equal to the intersection of the sets defined by φ_1 and φ_2.

(b). The subset of M^n defined by $\varphi_1(\vec{x}) \vee \varphi_2(\vec{x})$ is equal to the union of the sets defined by φ_1 and φ_2.

(c). The subset of M^{n-1} defined by $\exists x_n \varphi_1(x_1, \ldots, x_n)$ is equal to the projection of the set defined by φ_1 onto the first $n-1$ coordinates. In other words, let $\pi : M^n \rightarrow M^{n-1}$ map a tuple $(m_1, \ldots, m_n)$ to $(m_1, \ldots, m_{n-1})$. You are supposed to prove the set defined by $\exists x_n \varphi_1(x_1, \ldots, x_n)$ is equal to the image of the set defined by φ_1 under π.

Problem 5

(a). Fix L a countable language, and T a complete theory in L. Show that there can be no more than $2^{\aleph_0}$ many types. (Here $2^{\aleph_0}$ is the name for the size of the real numbers. It doesn't come up in the exam, but $\aleph_0$ is the name for the size of the natural numbers.)

(b). Now let $T = Th(\mathbb{N}, +, 0)$. Show that, for any prime number p, one can express "x is divisible by p" as a formula.

(c). Show that with T as in part (b), there are $|2^{\aleph_0}|$ types. That is show that there are $2^{\aleph_0}$ sets of formulas that are consistent.

(d). Show that there must be at least $|2^{\aleph_0}|$ non-isomorphic countable models of $Th(\mathbb{N}, +, 0)$.

Problem 6

Fix an alphabet A. For any subset W of A^*, let W' (pronounced "W jump" for some reason) be the set of oracle register machines with oracle W that halt on empty input. For example, if $0 \subset A^*$ is any decidable set, $0'$ is $\Pi_{halt} := \{w_p | P : \Box \to halt\}$.

(a). Show that for any $B \subset A^*$, $B' >_T B$. (Hint: For once, you're not going to reduce to the case of the halting problem. Instead, imitate it's proof).

(b). Show any enumerable set is oracle-decidable by some P using $0'$ as an oracle.