EXAM 1 – TAKEHOME SECTION

Problem 1: At various times in the class, we have used without proof the fact that any natural number has a unique prime factorization. Now, prove an easier statement: that every natural number can be written as a product of a power of two, and an odd number.

Problem 2: Recall that $\mathbb{N}_m$ is the set $\{n \in \mathbb{N} : n \leq m\}$. Define a finite sequence of natural numbers to be a function $f : \mathbb{N}_m \to \mathbb{N}$. Show that the set of all finite sequences of natural numbers is countable.

Problem 3: Recall that we define $\mathbb{C}$, the complex numbers, as follows: $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$. Moreover, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 + z_2$ is defined as $(x_1 + x_2) + i(y_1 + y_2)$ and we define $z_1 z_2 := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

(a). Prove that $\mathbb{C}$ is a field. That is, show that A1 to A4, M1 to M4, and D on page 23 are true of $\mathbb{C}$.

(b). Prove that $\mathbb{C}$ is not an ordered field. That is, prove that there is no subset $\mathbb{P}$ of $\mathbb{C}$ such that $\mathbb{P}$ satisfies conditions (i) to (iii) on page 25.

Problem 4. Define an infinite sequence of natural numbers to be a function $f : \mathbb{N} \to \mathbb{N}$. Show that the set of all infinite sequences of natural numbers is uncountable.

Problem 5. A point $x \in \mathbb{R}$ is said to be a boundary point of $A \subseteq \mathbb{R}$ iff every ϵ-neighborhood of x contains both points from A and points from the complement of A.

(a). Show that a set A and its complement have the same boundary points.

(b). Show that a set G is open if and only if it does not contain any of its boundary points.

(c). Show that a set F is closed if and only if it does contains all of its boundary points.

Problem 6. Suppose $a > 0$. Define a sequence inductively as follows: let $x_0 = a$, let $x_{n+1} = x_n - \sqrt{x_n} + 1$.

(a). Prove that (x_n) converges.

(b). Find $\lim(x_n)$.

Problem 7. Let $(x_n) \subset \mathbb{R}$. Show that the following are equivalent.

(1) $\lim(x_n) = x$

(2) For each open set $U \subseteq \mathbb{R}$ such that $x \in U$, there is a $K_U \in \mathbb{N}$ such that for $n > K_U$, $x_n \in U$

(3) For each $\epsilon > 0$, there is an m_ϵ such that each element of the m_ϵ-tail of (x_n) lies within distance ϵ of x.