EXAM 1 – TAKEHOME SECTION

Problem 1: At various times in the class, we have used without proof the fact that any natural number has a unique prime factorization. Now, prove an easier statement: that every natural number can be written as a product of a power of two, and an odd number.

Proof. This can be done either by induction or by the Well-Order Property of the natural numbers. I used well-ordering in class; I'll use (strong) induction here.

First note that any odd number, m, can be written a $2^0 \cdot m$, so it remains to show that any even number can be written in the desired fashion. Now we proceed by induction to show that $2k$ can be written in the desired form for every k in $\mathbb{N}$. The base case, $k = 1$, is clear. So now we assume that it is true for $k \leq m$, and prove it for $k = m + 1$.

So we want to show that $2(m+1)$ can be written as a product of a power of two, and an odd number. If $m + 1$ is odd, then we are done. So we may assume that $m+1$ is even, and equal to $2n$ for some $n \leq m$. Thus, by induction, $m + 1 = 2^j \cdot d$, for some odd number d. And thus, $2(m + 1) = 2^{j+1} \cdot d$, and we are done. □

Problem 2: Recall that $\mathbb{N}_m$ is the set $\{n \in \mathbb{N} : n \leq m\}$. Define a finite sequence of natural numbers to be a function $f : \mathbb{N}_m \to \mathbb{N}$. Show that the set of all finite sequences of natural numbers is countable.

Proof. First we note that a function from $\{1, \ldots, m\}$ to $\mathbb{N}$ can be written as $(n_1, \ldots, n_m)$, and conversely an m-tuple $(n_1, \ldots, n_m)$ can be thought of as the function that maps 1 to n_1, 2 to n_2, etc. Thus the set of functions from $\mathbb{N}_m$ to $\mathbb{N}$ is in bijection with the set of m-tuples, $\mathbb{N} \times \ldots \times \mathbb{N}$. Thus the set of all finite sequences of natural numbers is in bijection with $\mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N} \times \mathbb{N}) \cup \ldots$ and we just have to show that each of these sets is countable.

Obviously $\mathbb{N}$ is countable, and the it is proven in the textbook that $\mathbb{N} \times \mathbb{N}$ is countable. It is easy to see whenever A and B are countably infinite that there is a bijection between $A \times B$ and $\mathbb{N} \times \mathbb{N}$. Thus $A \times B$ is countably infinite. Then mathematical induction shows that $\mathbb{N} \times \ldots \times \mathbb{N}$ is always countable, and we are done. □

Problem 3: Recall that we define $\mathbb{C}$, the complex numbers, as follows: $\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$. Moreover, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then $z_1 + z_2$ is defined as $(x_1 + x_2) + i(y_1 + y_2)$ and we define $z_1 z_2 := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

(a). Prove that $\mathbb{C}$ is a field. That is, show that A1 to A4, M1 to M4, and D on page 23 are true of $\mathbb{C}$.

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Proof. I meant this mostly as a reminder about how the complex numbers work. It shouldn't have been hard. I'll write the proof for A1, as an examples, and skip the rest. (You don't get to do this on your exams! However, it wasn't the clearest problem, and I'm going to be extremely generous in grading it.)

To show A1, we have to show that for any $z_1, z_2, \in \mathbb{C}$, $z_1 + z_2 = z_2 + z_1$. Let $z_1 = x_1 + iy_1$ and let $z_2 = x_2 + iy_2$. Then $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) = (x_2 + x_1) + i(y_2 + y_1) = (x_2 + iy_2) + (x_1 + iy_1)$. □

(b). Prove that $\mathbb{C}$ is not an ordered field. That is, prove that there is no subset $\mathbb{P}$ of $\mathbb{C}$ such that $\mathbb{P}$ satisfies conditions (i) to (iii) on page 25.

Proof. We do a proof by contradiction in two cases:

Case 1: i is in $\mathbb{P}$: Then $i^2 = -1$ is in $\mathbb{P}$, and thus $-1 \cdot i = -i$ is in $\mathbb{P}$. But this contradicts the fact that only one of i or $-i$ can be in $\mathbb{P}$.

Case 2: i is not in $\mathbb{P}$: Then $(-i)^2 = -1$ is in $\mathbb{P}$. But just as above, this implies that i is also in $\mathbb{P}$ a contradiction. □

Problem 4. Define an infinite sequence of natural numbers to be a function $f : \mathbb{N} \to \mathbb{N}$. Show that the set of all infinite sequences of natural numbers is uncountable.

Proof. This proof works like the 2nd proof that the real numbers are uncountable, the one that uses decimal notation. My write-up isn't as nicely formated, so it might be useful to look at the write up of Theorem 2.5.5, if this isn't clear. Suppose we have an enumeration, $x_1, x_2, x_3, \ldots$, of all infinite sequences of natural numbers. We will get a contradiction by creating an infinite sequence, y, not listed in the enumeration. Suppose that the sequence $x_1 = (n_1, n_2, n_3, \ldots)$. That is, suppose that as a function $x_1(1) = n_1$, $x_1(2) = n_2$, and so on. Then our new sequence will start out $y = (n_1) + 1, \ldots)$. That is, we let $y(1) = x_1(1) + 1$. In general, we let $y(n) = x_n(n) + 1$. Thus y cannot be equal to x_i for any i, a contradiction.

□

Problem 5. A point $x \in \mathbb{R}$ is said to be a boundary point of $A \subseteq \mathbb{R}$ iff every ϵ-neighborhood of x contains both points from A and points from the complement of A.

(a). Show that a set A and its complement have the same boundary points.

Proof. We simply write down the definition of what it means to be a boundary point of A^C, the complement of A: A point $x \in \mathbb{R}$ is said to be a boundary point of A^C iff every ϵ-neighborhood of x contains both points from A^C and points from the complement of A^C. But since the complement of A^C is just A, this is the same as the definition of a boundary point of A. □

(b). Show that a set G is open if and only if it does not contain any of its boundary points.

Proof. Suppose G is open and $x \in G$. Then there is an ϵ-neighborhood of x contained in G. Since it's contained in G, it can't contain any points of G^C, and hence x is not a boundary point.

Now suppose that G does not contain any of its boundary points. Now take y in G. Since y is not a boundary point, it is not true that every ϵ-neighborhood of y contains both points from G and points from G^C. Thus it is true that for some ϵ,

$V_\epsilon(y)$ contains only points from G, or it contains only points from G^C. But $V_\epsilon(y)$ contains y, and $y \in G$, so it is not possible for $V_\epsilon(y)$ to contain points only from G^C. Thus $V_\epsilon(y) \subseteq G$. But we have taken an arbitrary point in G, and shown that some ϵ-neighborhood of that point is contained in G, and this is what it means to be an open set. □

(c). Show that a set F is closed if and only if it does contains all of its boundary points.

Proof. Suppose that F is closed. Then F^C is open, and does not contain any of its boundary points by (b). So all of the boundary points of F^C are in F. But, by (a), the boundary points of F^C are precisely the boundary points of F.

Now suppose that F contains all of its boundary points. So F^C contains none of its boundary points. So F^C is open. Thus F is closed. □

Problem 6. Suppose $a > 0$. Define a sequence inductively as follows: let $x_0 = a$, let $x_{n+1} = x_n - \sqrt{x_n} + 1$.

(a). Prove that (x_n) converges.

(b). Find $\lim(x_n)$.

Problem 7. Let $(x_n) \subset \mathbb{R}$. Show that the following are equivalent.

(1) $\lim(x_n) = x$

(2) For each open set $U \subseteq \mathbb{R}$ such that $x \in U$, there is a $K_U \in \mathbb{N}$ such that for $n > K_U$, $x_n \in U$

(3) For each $\epsilon > 0$, there is an m_ϵ such that each element of the m_ϵ-tail of (x_n) lies within distance ϵ of x.