EXAM 2 – TAKEHOME SECTION

Problem 1:

(a). Take $m \in \mathbb{N}$. Show directly (that is, from the definitions) that $\left((m^n)^{1/n}\right)$ converges, and find its limit. (Yes, this is easy.)

(b). Take $k \in \mathbb{N}$. Show that $\left((k)^{1/n}\right)$ converges and find its limit. (Also easy.)

(c). Take $k, m \in \mathbb{N}$. Show that $\left((k(m^n))^{1/n}\right)$ converges and find its limit. (Still not so hard.)

(d). Let $m_1, \ldots, m_k$ be elements of $\mathbb{N}$. Show that $\left((m_1^n + \ldots + m_k^n)^{1/n}\right)$ converges and find its limit. (Requires some thought.)

Problem 2:

(a). Let (x_n) be a bounded sequence, and for each $n \in \mathbb{N}$ let $s_n :=$sup$\{x_k : k \geq n\}$, and let $t_n :=$inf$\{x_k : k \geq n\}$. Prove that (s_n) and (t_n) are monotone and convergent. We call lim(s_n) the limit superior of (x_n), or limsup(x_n), and we call lim(t_n) the limit inferior of (x_n) or liminf(x_n).

(b). Prove that if limsup(x_n) =liminf(x_n), then (x_n) is convergent.

(c). Give an example of a convergent sequence (x_n) such that for all m and k, $s_m \neq x_k$. (Yes, this is easy.)

(d). Show that if (x_n) is convergent then $\limsup(x_n) = lim(x_n)$.

(e). Say (a_n) and (b_n) are bounded sequences. Show that $\limsup(a_n + b_n)$ exists and that $\limsup(a_n + b_b) \leq \limsup(a_n) + \limsup(b_n)$.

(f). Find an example where the inequality above is strict.

Problem 3. Find the limits of the following sequences: (Hint: the answers are $1/e$, e, and $e^{3/2}$. A second hint: looking at Example 3.3.6 may be helpful.)

(a). (x_n), where $x_n := \left(\frac{n}{n+1}\right)^n$

(b). (y_n), where $y_n := (1 + 1/n)^{n+1}$

(c). (z_n), where $z_n := (1 + 1/2n)^{3n+1}$

Problem 4: Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences than so are:

(a). $(x_n + y_n)$

(b). $(x_n \cdot y_n)$

Problem 5.

(a). Let $f : \mathbb{R} \to \mathbb{R}$, let I be an open interval, and let $c \in I$. Let f_1 be the restriction of f to I. Show that f_1 has a limit at c iff f has a limit at c.

(b). Let $f : \mathbb{R} \to \mathbb{R}$, let I be an closed interval, and let $c \in I$. Let f_2 be the restriction of f to I. Show that if f has a limit at c then so does f_2.

(c). Let f_2, c, and I be as above. Show that f_2 may have a limit at c even though f does not. (And go back to make sure that your proof in part (a) doesn't work for I closed.)

Problem 6: Here we prove a couple of equivalent definitions of continuity.

(a). Prove Theorem 5.1.2.

(b). If a function, f, from $\mathbb{R}$ to $\mathbb{R}$ is continuous, then for any open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is an open set.

(c). If a function, f, from $\mathbb{R}$ to $\mathbb{R}$ has the property that for any open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is an open set, then f is continuous.

(d). Give a example of a continuous function, and an open interval I, such that $f(I)$ is not itself an open set.

(e). Prove that a function is continuous iff the inverse image of every closed set is closed.

Problem 7: This is something that I talked about in class but never wrote down. It's worth emphasizing. Say a point $c \in A$ is an *isolated* point of A iff there is some $V_\delta(c)$ such that c is the only point of A contained in $V_\delta(c)$. Show c is an isolated point of A if and only if it is not a cluster point of A